AD-A081 382 ROCHESTER UNIV N Y GRADUATE SCHOOL OF MANAGEMENT F/G 4/2
GAUSSIAN MARKOV RELATED VARIATES FOR METEOROLOGICAL PLANNING.(U)
NOV 79 J KEILSON, H F ROSS F19628-78-C-0158
UNCLASSIFIED AFGL-TR-79-0282 NL

1 OF 1
ADA
081382

END
DATE
FILMED
4-80
DTIC

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
18 1. REPORT NUMBER AFGL-TR-79-0282	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9
6 4. TITLE (and Subtitle) GAUSSIAN MARKOV RELATED VARIATES FOR METEOROLOGICAL PLANNING.		5. TYPE OF REPORT & PERIOD COVERED Final Report. 1 July 78 — 30 Sep 79
		6. PERFORMING ORG. REPORT NUMBER
10 7. AUTHOR(s) J. Keilson H. F. Ross		8. CONTRACT OR GRANT NUMBER(s) 15 F19628-78-C-0158
9. PERFORMING ORGANIZATION NAME AND ADDRESS ~~The~~ Graduate School of Management Univ~~ersity~~ of Rochester Rochester, New York 14627		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62101F 16 667009AE 17 04
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Monitor/Irving I. Gringorten/LYD	11	12. REPORT DATE 14 Nov[redacted]79
		13. NUMBER OF PAGES 60
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 59		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Gaussian-Markov process
Ornstein-Uhlenbeck process
Meteorology
Fluctuations
Extreme values
Interval maximum

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Gaussian-Markov process or Ornstein-Uhlenbeck (O-U) process successfully models fluctuations in upper-level wind velocity, rainfall, temperatures, etc. ~~(Gringorten, 1966, 1968).~~ On-line predictions for long-range operational planning require the distribution of the maximum value of X(t), the stationary O-U process, over an interval of specified duration. The required distribution and its basic properties are presented with accompanying tables and graphs. Algorithmic techniques required for on-line predictions are indicated.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

SECURITY CLASSIFICATION OF THIS PAGE(*When Data Entered*)

SECURITY CLASSIFICATION OF THIS PAGE(*When Data Entered*)

Final Report

Gaussian Markov Related Variates for Meteorological Planning

Table of Contents

INTRODUCTION

It has been observed empirically (Gringorten, 1966, 1968) that fluctuations in upper-level wind velocity, rainfall, temperature, and other meteorological variates may be modeled by a type of random Markov process. A good fit is obtained over a broad range of values when the so-called Ornstein-Uhlenbeck process described below is employed. Acceptance of the validity of such a model permits utilization of its theoretical predictions for long-range operational planning, with on-line predictions available from software. Such software is in development at the U.S.A.F. Geophysical Research Laboratory.

The O-U process is the simplest random process having a stationary distribution whose sample paths are continuous functions of time. It is correspondingly easiest to work with. Certain passage time distributions needed for meteorological planning have been difficult, however, to calculate, and have only recently become available (Keilson and Ross, 1975).

The stationary Gaussian Markov process or O-U process $X(t)$ has the following characteristics

(a) $E[X(t)] = 0$

(b) $\mathrm{Var}\, X(t) = 1$

(c) $\mathrm{cov}[X(t),\, X(t + \tau)] = e^{-|\ \ |}$

(d) The probability density function is standard normal, i.e.,

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

(e) The transition probability cumulative distribution function is

$$P[X(t) \leq x \mid X(0) = x_0] = \int_{-\infty}^{x} \frac{\exp\{-(y - x_0 e^{-t})^2 / 2(1 - e^{-2t})\}}{\sqrt{2\pi(1 - e^{-2t})}} dy$$

Many related variates are of meteorological interest. Of central interest to this study has been the maximum value of $X(t)$ attained over an interval of specified duration θ, i.e.,

$$M(\theta) = \max_{t \leq t' \leq t+\theta} X(t) \quad .$$

The distribution of $M(\theta)$ relates directly to the passage time distributions previously obtained (Keilson-Ross, 1975). The relationship, though direct and straightforward in principle, has required extensive analytical and numerical effort to convert to useful form. The results needed have been attained. A concise presentation of the results was given at Banff, Canada in October, 1979 at the Sixth Conference on Probability and Statistics in Atmospheric Sciences. The paper given there is Part II of this final report.

The Banff paper of Part II is a précis of a longer paper documenting our results in full detail with complete tables and graphs given. This longer paper is Part I of this final report.

The programs needed to make the numerical results available on-line are available, and will be developed for USAF use at the Air Force Geophysics Laboratory (AFSC) at Hanscom Air Force Base in Massachusetts.

References

Gringorten, I. I. (1966). "A stochastic model of the frequency and duration of weather events," J. App. Meteorology, 5, pp. 606-624.

Gringorten, I. I. (1968). "Estimating finite-time maxima and minima of a stationary Gaussian Ornstein-Uhlenbeck process by Monte Carlo simulation," Amer. Stat. Assn. J., pp. 1517-21.

Keilson, J. and Ross, H. F. (1975). "Passage time distributions for Gaussian Markov statistical processes," Selected Tables in Mathematical Statistics, Vol. III (edited by the Institute of Mathematical Statistics). American Math. Society, Providence, Rhode Island.

Keilson, J. and Ross, H. F. (1978). "The maximum of the stationary Gaussian Markov process over an interval," Graduate School of Management, University of Rochester, Working Paper Series No. 7910.

Keilson, J. and Ross, H. (1979). "The Maximum over an interval of meteorological variates modeled by the stationary Gaussian Markov process," Proceedings of the Sixth Conference on Probability and Statistics in Atmospheric Sciences, Banff, Alberta, Canada, October 9-12, pp. 213-216.

The University of Rochester
Graduate School of Management

THE MAXIMUM OF THE STATIONARY GAUSSIAN MARKOV PROCESS OVER AN INTERVAL

Part I

J. Keilson
University of Rochester

H. F. Ross
University of Birmingham

August, 1978

This work was supported by the U.S. Air Force Geophysics Laboratory under Contract F19628-78-C-0158.

Table of Contents

Tables and Graphs

1. Introduction and Summary

The Gaussian Markov process, also known as the Ornstein-Uhlenbeck process, has been widely employed to describe fluctuations in physical phenomena. It has also been a basic theoretical and computational tool for statistics. References to such O-U applications have been presented in [10], where O-U passage time distributions were described and tabulated.

An important potential application of the O-U process is to meteorological prediction and operational planning. I. Gringorten has reported extensively [8,9] on the empirical validity of the O-U process as a model for fluctuations in temperature and other climatic variates. For meteorological applications, the range of such variates and the persistence of high or low values of such variates are of considerable importance. This report is concerned with the maximum of such variates over a specified time interval predicted by an Ornstein-Uhlenbeck model for the fluctuations. More precisely, we are interested in the maximum value $M(\theta)$ of the process over an interval of length θ, when the process has its stationary distribution, the standard normal distribution, at the beginning of the interval. We study, therefore, the sequence of random variables

$$(1) \qquad M(\theta) = \max_{0 \le t' \le \theta} X(t')$$

where $X(t)$ is the stationary O-U process with distribution

$$(2) \qquad F_\theta(x) = P[X(t) \le x] = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy \quad , \qquad 0 \le t < \infty \quad ,$$

and covariance function

$$r_X(t) = E[X(t)X(t+\tau)] = e^{-\tau} \quad . \tag{3}$$

Physical processes of interest described by an O-U model have three parameters of interest: the mean value m, the rms amplitude A, and the time scale. A process V(u) with mean m, amplitude A and correlation function $\rho_V(\tau) = \mathrm{cov}[V(u), V(u+\tau)]/A^2 = e^{-\tau/T}$ would be described by

$$V(u) = m + AX(u/T) \tag{4}$$

where X(t) is our O-U process. We note that both the process X, its states x and time parameter t are dimensionless. The O-U process has zero mean and standard deviation (amplitude) equal to one.

The distribution of the maximum $M(\theta)$ for interval length θ is related to the passage time distributions previously described [10]. The calculation technique employed there evaluated zeros and residues in the complex plane of the parabolic cylinder functions. Such zero and residue evaluation techniques again feature in tabulating the distribution of $M(\theta)$.

The extreme values of $M(\theta)$ for large θ have been described in indirect form by Darling and Erdos in the form of a limit theorem [4]. The appropriate form of the limit theorem is presented in Section 3. Such limit theorem information has been of limited value for prediction and planning because such limit theoretic behavior has been known to set

in quite slowly, and when the limit theorem was useful has not been known. Correspondingly, the error in the limiting distribution has not been known for any value of θ.

The distribution of $M(\theta)$ and the underlying theory supporting its tabulation are given. The distribution $F_\theta(y)$ of $M(\theta)$ and the associated density $f_\theta(y)$ are tabulated and plotted graphically. Comparison with the limit theorem of Darling and Erdos is discussed.

A basic feature of the behavior of $M(\theta)$ is that for large values of θ, $M(\theta)$ loses its variability. Formally, one has

$$M(\theta) \xrightarrow{p} \sqrt{2\log\theta} \quad , \tag{4}$$

i.e., $M(\theta)$ converges in probability (or distribution) to the constant $\sqrt{2\log\theta}$. Such behavior is familiar in extreme value theory, accounts of which may be found in [7] and [1].

2. Basic Theory

A. The distribution of the maximum and its density

Let $X(t)$ be the stationary O-U process [3] for which the density of $X(t)$ is standard normal, i.e.,

$$(1) \qquad f_{X(t)}(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

and the covariance function is

$$(2) \qquad r_X(\tau) = \mathrm{cov}[X(t), X(t+\tau)] = e^{-\tau} \quad .$$

Let $M(\theta)$ be the maximum of $X(t)$ over an interval of length θ, and let $F_\theta(y)$ be its c.d.f., i.e.,

$$(3) \qquad F_\theta(y) = P[M(\theta) < y] = P[X(t') < y,\ 0 \le t' \le \theta] \ , \quad -\infty < y < \infty \quad .$$

We wish to evaluate the distribution $F_\theta(y)$ and its p.d.f.

$$(4) \qquad f_\theta(y) = \frac{d}{dy} F_\theta(y) \ , \quad -\infty < y < \infty \quad .$$

We also seek to quantify the asymptotic behavior of $F_\theta(y)$ and $f_\theta(y)$ as $\theta \to \infty$, and compare our results with the limit form of this distribution of extreme values, given effectively by Darling and Erdos [4].

The methods employed in [10] to calculate the passage time distributions for the O-U process are applicable with some extension to the calculation

of the distribution of $M(\theta)$. The desired variate $M(\theta)$ is related to the <u>stationary</u> O-U process, and this feature of stationarity must be dealt with.

To understand the relation of the passage time problem to the maximum problem, a discussion in the context of a simpler birth-death process will be helpful.

Consider a birth-death process $N(t)$ assumed to be stationary on states $\{0,1,2,\ldots,K\}$. Let $M(\theta) = \max_{0 \le t' \le \theta} N(t')$. Then if T_{rn} is the time to go from state r to state n and if e_r is the probability that the stationary $N(t)$ is in state r, one has

$$P[M(\theta) \ge n] = \sum_{r=0}^{n-1} P[T_{rn} < \theta]e_r + \sum_{r=n}^{K} e_r \tag{5}$$

$$= 1 - \sum_{r=0}^{n-1} e_r P[T_{rn} > \theta] \quad ,$$

so that

$$P[M(\theta) < n] = \sum_{r=0}^{n-1} e_r P[T_{rn} > \theta] \quad . \tag{6}$$

Let $E[e^{-sT_{rn}}] = \sigma_{rn}(s) = \int_0^\infty e^{-s\tau} s_{rn}(\tau)d\tau$ where $s_{rn}(\tau)$ is the p.d.f. of T_{rn}. Let $\bar{S}_{rn}(\tau) = P[T_{rn} > \tau]$ be the survival function of T_{rn}. Then one has $\int_0^\infty e^{-s\tau} P[T_{rn} > \tau]d\tau = [1 - \sigma_{rn}(s)]/s$. Hence from (6) we have

$$\int_0^\infty e^{-s\theta} P[M(\theta) < n]d\theta = \sum_{r=0}^{n-1} e_r \left\{\frac{1 - \sigma_{rn}(s)}{s}\right\} \quad . \tag{7}$$

The O-U process may be regarded as the limit of a sequence of birth-death processes and the structure of equation (7) then carries over directly to give

$$\int_0^\infty e^{-s\theta} P[M(\theta) < y] d\theta = \int_{-\infty}^{y} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left\{\frac{1 - \sigma_{xy}(s)}{s}\right\} dx \tag{8}$$

where $M(\theta)$ is the stationary maximum for the O-U process and $\sigma_{xy}(s)$ is the Laplace transform of the passage time density from x to y. It has been shown, however, in [5] that

$$\sigma_{xy}(s) = \frac{D_{-s}(-x)}{D_{-s}(-y)} \exp[(x^2 - y^2)/4] \tag{9}$$

where $D_\nu(z)$ is the Weber function of order ν ([12], p. 323, et seq.). Equations (8) and (9) are the basic equations for our calculations. As shown in Appendix 1, the function $D_{-s}(-y)$ is an entire function of s for all y, with zeros only on the negative real s axis. These zeros are simple and give rise to simple poles for the expression in (9), and the locations of the poles and residues contributing to $P[M(\theta) < y]$ for the inversion of (8) are found as in [10]. It is shown in Appendix 1 that

$$\int_{-\infty}^{y} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sigma_{xy}(s) dx = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \frac{D_{-s-1}(-y)}{D_{-s}(-y)} = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\beta_n(y)}{s + \lambda_n(y)} \tag{10}$$

where $\lambda_n(y)$ and $\beta_n(y)$ are positive and $0 < \lambda_1(y) < \lambda_2(y) \ldots$. From (8) one then has (Appendix 1)

$$F_\theta(y) = P[M(\theta) < y] = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\beta_n(y)}{\lambda_n(y)} e^{-\lambda_n(y)\theta} \tag{11}$$

This may be rewritten as

$$F_\theta(y) = F_0(y) \sum_{n=1}^{\infty} p_n(y) e^{-\lambda_n(y)\theta} \tag{12}$$

where $F_0(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, and

$$p_n(y) = \frac{e^{-y^2/2}/\sqrt{2\pi}}{F_0(y)} \frac{\beta_n(y)}{\lambda_n(y)} \tag{13}$$

B. <u>The ergodic exit time density $s_y^E(\tau)$</u>

The integral appearing in (10) relates to a simple entity of probabilistic interest, $s_y^E(\tau)$, the "ergodic exit time density" from $(-\infty,y)$ [11]. This is the p.d.f. of the time to depart from $(-\infty,y)$ if one starts with the stationary distribution truncated to $(-\infty,y)$ and renormalized to unity. Thus $s_y^E(\tau) = -\frac{d}{d\tau} P[X(t') \le y,\ 0 \le t' \le \tau | X(0) \le y]$. Its Laplace transform is then given by

$$\sigma_y^E(s) = \{F_0(y)\}^{-1} \int_{-\infty}^{y} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \sigma_{xy}(s) dx \tag{14}$$

It is known that Markov diffusion processes are reversible in time and that $s_y^E(\tau)$ for such processes is completely monotone in τ [11]. It then follows that the integral on the right hand side of (8) is equal to $F_0(y)\bar{S}_y^E(\tau)$ where $\bar{S}_y^E(\tau) = \int_\tau^\infty s_y^E(t')dt'$ is the survival function of the ergodic exit

time and is also completely monotone in τ. Hence $p_n(y)$ must be positive, and this positivity is borne out by the computation. We note from (12) that $\sum_{n=1}^{\infty} p_n(y) = 1$. This, too, emerges from the computation and provides an accuracy check.

The density of $M(\theta)$, $f_\theta(y)$ may then be obtained from (12) via numerical differentiation. Alternately, (Appendix 1) one finds analytically that

$$(15) \qquad L_s[f_\theta(y)] = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \left[\frac{D_{-s-1}(-y)}{D_{-s}(-y)}\right]^2 = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \left[\sum_1^{\infty} \frac{\beta_n(y)}{s + \lambda_n(y)}\right]^2$$

where $L_s[f_\theta(y)] = \int_0^{\infty} \exp(-s\theta) f_\theta(y)d\theta$. Hence, from (10) and (14), we have

$$(16) \qquad f_\theta(y) = \sqrt{2\pi}\, e^{\frac{1}{2}y^2}\, F_0^2(y)\, \{s_y^E(\theta) \underset{\theta}{*} s_y^E(\theta)\} \quad .$$

We note from (15) that $\lim_{s\to\infty} sL_s(f_\theta(y)) = f_0(y) = e^{-y^2/2}/\sqrt{2\pi}$ implies that

$$(17) \qquad \sum_1^{\infty} \frac{\beta_n(y)}{s + \lambda_n(y)} \sim \frac{1}{\sqrt{s}}, \quad s \to \infty \quad .$$

Since from (10), (14), and (17),

$$L_s[s_y^E(\theta)] = \sigma_y^E(s) = \frac{e^{-y^2/2}/\sqrt{2\pi}}{F_0(y)} \left[\sum_1^{\infty} \frac{\beta_n(y)}{s + \lambda_n(y)}\right]$$

$$\sim \frac{e^{-y^2/2}/\sqrt{2\pi}}{F_0(y)} \frac{1}{\sqrt{s}} \quad , \quad s \to \infty$$

one has

$$(18)\qquad s_y^E(\theta) \sim \frac{e^{-y^2/2}}{\sqrt{2\pi}\,F_0(y)}\,\frac{1}{\sqrt{\pi}}\,\frac{1}{\sqrt{\theta}}\ , \qquad \theta \to 0+ \quad .$$

C. An arc sine law for $F_\theta(0) = P[M(\theta) < 0]$

The state $x = 0$ is a special state for the O-U process, and many of the descriptive densities of the process take a simpler form when this state is involved. The c.d.f. $F_\theta(y)$ and the density $f_\theta(y)$ also take simpler form for $y = 0$, and the closed form answers obtained are structurally informative. Details are provided in Appendix 2.

The ergodic exit time density $s_0^E(\tau)$ from the interval $(-\infty,0)$ is given (Appendix 2) by

$$(19)\qquad s_E(\tau) = \frac{2}{\pi}\, e^{-\tau}(1 - e^{-2\tau})^{-1/2} \quad .$$

This may be rewritten as

$$(20)\qquad s_E(\tau) = \sum_{n \text{ odd}} p_n\, n e^{-n\tau} \quad ,$$

where $p_n = p_n(0)$ (cf. Eq. (3.8)) is the value of our basic positive weights $p_n(y)$ at $y = 0$. The p_n are shown in the appendix to be given explicitly by

$$(21a)\qquad p_{2k} = p_{2k}(0) = 0 \quad ;$$

$$(21b)\qquad p_{2k+1} = p_{2k+1}(0) = \frac{2}{\pi}\,\frac{\Gamma(k + 1/2)}{\Gamma(1/2)}\,\frac{1}{k!\,(2k + 1)} \quad .$$

One then has, asymptotically, for the slowly convergent sequence p_{2k+1},

(22) $$p_{2k+1} \sim (\pi n)^{-3/2} \quad , \quad n \to \infty \quad .$$

We note from (20) that the ergodic exit time T_0^E has all moments. We also see that $s_E(\tau)$ is completely monotone in τ. A direct integration of (19) gives

(23) $$\bar{S}_0^E(\tau) = \frac{2}{\pi} \arcsin(e^{-\tau}) \quad ,$$

an arcsin law for T_0^E. It then follows directly from (23) nd (3.7) that

(24) $$F_\theta(0) = P[M(\theta) < 0] = \frac{1}{\pi} \arcsin e^{-\theta} \quad .$$

Hence, one has the asymptotic behavior

(25) $$P[M(\theta) < 0] \sim \frac{1}{\pi} e^{-\theta} \quad , \quad \theta \to \infty \quad .$$

3. The Asymptotic Distribution of $M(\theta)$

A. The Basic Limit Theorem

It may be inferred (see Appendix 3) from the ideas and results in Darling and Erdos [4] that the interval maximum $M(\theta)$ satisfies the limit theorem

$$(1) \qquad \lim_{\theta\to\infty} P[c(\theta)\{M(\theta) - \zeta(\theta)\} < x] = \exp\{-e^{-x}\} = G(x)$$

where

$$(2a) \qquad c(\theta) = \sqrt{2\log\theta} \quad ,$$

and

$$(2b) \qquad \zeta(\theta) = c(\theta) + \frac{\log c(\theta)}{c(\theta)} - \frac{\log\sqrt{2\pi}}{c(\theta)} \quad .$$

B. Validity of the Limit Distribution

The appearance of the extreme value distribution $G(x)$ in (1) is to be expected, of course, from the more basic results of extreme value theory as developed, for example, by Gnedenko [6] and explored statistically by Gumbel [7]. Indeed, the nature of the stationary O-U process is such that the sequence of values $(X(ak),\ k = 0,1,2,\ldots)$ are normally distributed and asymptotically independent as $a \to \infty$. This, of course, is true for any stationary Gaussian process $Y(t)$ whose covariance function $r_Y(\tau)$ vanishes for large τ. It might then be anticipated that the extreme value theory developed for discrete sequences of independent random variables is still valid for the continuous time O-U process. The Darling-Erdos

theory provides the proof that this is so. It does not, however, say for what values of θ and y the approximation

$$(4) \qquad F_\theta(y) = P[M(\theta) < y] \approx G(c(\theta)y - c(\theta)\zeta(\theta))$$

implicit in the limit theorem of Eq. (1) is valid. It also provides no information on the error contained in the approximation (4). The tabulated values of $F_\theta(x)$ and associated graphs are therefore essential for prediction and planning.

C. "Deterministic" Behavior of $M(\theta)$ for Large θ

Since $c(\theta) \to \infty$, as $\theta \to \infty$, and $c(\theta) - \zeta(\theta) \to 0$ as $\theta \to \infty$, it follows from (1) that

$$\lim_{\theta\to\infty} P[|M(\theta) - c(\theta)| < \epsilon] = 0 \quad , \quad \text{all } \epsilon > 0 \quad ,$$

i.e., that

$$(5) \qquad M(\theta) - c(\theta) \overset{P}{\to} 0 \quad , \quad \theta \to \infty \quad .$$

This implies that $M(\theta)$ converges in distribution to the constant $c(\theta)$ as $\theta \to \infty$, i.e., that $M(\theta)$ loses its uncertainty asymptotically. One also expects that $M(\theta)$ converges in time to $c(\theta)$ for every sample path with probability one, i.e., that the strong law holds. One knows that $M(\theta)$

has all moments, and that the asymptotic law $G(x)$ for $c(\theta)\{M(\theta) - \zeta(\theta)\}$ has all moments. From this and the numerical information on $f_\theta(y)$ one expects that $M(\theta)$ will have all moments for all θ. A proof seems difficult. Since $\zeta(\theta)/c(\theta) \to 1$, as $\theta \to \infty$, $c(\theta)$ describes both the displacement of $M(\theta)$ from zero and the scale of the deviation of $M(\theta)$ from its mode as measured, e.g., by the half-width of $M(\theta)$ about its mode. The linear relation between the mode and such a scale factor is evident from the graph of the density $f_\theta(y)$.

A simple table of values for $c(\theta)$ and $\zeta(\theta)$ is shown in Figure 1, demonstrating the extreme slowness with which $c(\theta)$ increases with θ.

$c(\theta)$	$\zeta(\theta)$	θ
1	0.08	1.65
2	1.89	7.39
3	3.06	9.00×10^1
4	4.12	2.98×10^3
5	5.14	2.68×10^5
6	6.15	6.57×10^7

FIGURE 1

The mean and variance for the distribution function $G(x)$ may be obtained from its moment generating function (cf. [7])

(5a) $$\gamma(s) = \int_{-\infty}^{\infty} e^{+sx} dG(x) = \int_{-\infty}^{\infty} e^{sx} e^{-x} \exp\{-e^{-x}\} dx$$

$$= \int_0^{\infty} t^{-s} e^{-t} dt = \Gamma(1-s) \quad .$$

We then have $\mu_G = -\Gamma'(1) = 0.577215\ldots$ (Euler's constant), and from $\sigma_G^2 = [\log \gamma(s)]_0''$ we find $\sigma_G^2 = \pi^2/6$, i.e., $\sigma_G = 1.28255$. Equation (1) may be employed to write for $M(\theta)$ the approximate distribution valid for large θ

$$M(\theta) \overset{d}{\approx} \zeta(\theta) + \frac{\xi_G}{c(\theta)} \tag{5b}$$

where ξ_G has mean and variance μ_G, and σ_G^2 given above. We find, therefore, that for large θ

$$E[M(\theta)] \approx \zeta(\theta) + \mu_G/c(\theta) \ ; \tag{5c}$$

$$\sigma_{M(\theta)} \approx \frac{\sigma_G}{c(\theta)} \quad . \tag{5d}$$

The density $g(x)$ of ξ_G is $e^{-x}\exp\{-e^{-x}\}$ and has its mode at 0. It then follows from (5b) that for large θ the density $f_\theta(y)$ of $M(\theta)$ has its mode at $\zeta(\theta)$, i.e.,

$$y_{MODE}(\theta) \approx \zeta(\theta) \ , \quad \theta >> 1 \quad .$$

It is of interest to compare the asymptotic behavior of $M(\theta)$ with that of $M_k = \max_{1 \le j \le k} X_j$, where X_j are i.i.d. and standard normal. In that case, one finds that M_k converges in probability to $\sqrt{2\log k}$ with correction terms of a character similar in form to that for $M(\theta)$. Details may be found in Cramer [2] and in [1]. That M_k and $M(k)$ both go to $\sqrt{2\log k}$ in probability is in keeping with the fact that $M(\theta)$ for the O-U process has a "relaxation time" equal to unity.

D. Relation of the Asymptotic Distribution of $M(\theta)$ to $\lambda_n(y)$ and $p_n(y)$

For every value of y, one has the relation

$$(6) \qquad F_\theta(y) = P[M(\theta) < y] = P[M(0) < y]P[T_y^E > \theta]$$

i.e.,

$$(7) \qquad F_\theta(y) = F_0(y)\bar{S}_y^E(\theta)$$

where T_y^E is the ergodic exit time discussed in Section 2, part B. This may be inferred immediately from (2.8) and (2.14). For large y, $P[M(0) < y] \to 1$ and the asymptotic behavior of $M(\theta)$ is contained in the survival function $\bar{S}_y^E(\theta)$ of the ergodic exit time. From Equations (2.12) and (7) we have

$$(8) \qquad \bar{S}_y^E(\theta) = \sum_{n=1}^{\infty} p_n(y)e^{-\lambda_n(y)\theta} \quad .$$

A graph of the eigenvalues $\lambda_n(y)$ plotted against y is given in Figure 2. It is seen that all the eigenvalues $\lambda_n(y)$ decrease with y for all real y, and that for the principal eigenvalue $\lambda_1(y)$, one has

$$(9a) \qquad \lim_{y\to\infty} \lambda_1(y) = 0 \ ; \quad \lambda_1(0) = 1 \quad .$$

For the other eigenvalues, one finds

$$(9b) \qquad \lim_{y\to\infty} \lambda_n(y) = n-1 \ ; \quad \lambda_n(0) = 2n-1 \quad ,$$

and one notes that for $0 < y < \infty$, the values $\lambda_n(y)$ have approximately linear spacing (Fig. 2).

We have seen in Section 2 that $p_n(y) > 0$, and $\sum p_n(y) = 1$ for all y. It is also found numerically that $p_1(y)$, the mass associated with the maximal eigenvalue, increases monotonically with y for all real y and that

(10a) $$\lim_{y\to\infty} p_1(y) = 1 \quad .$$

Correspondingly, one finds

(10b) $$\lim_{y\to\infty} p_n(y) = 0 \quad , \quad n \geq 2 \quad .$$

A graph of $p_n(y)$ obtained from the computer is shown in Figure 3.

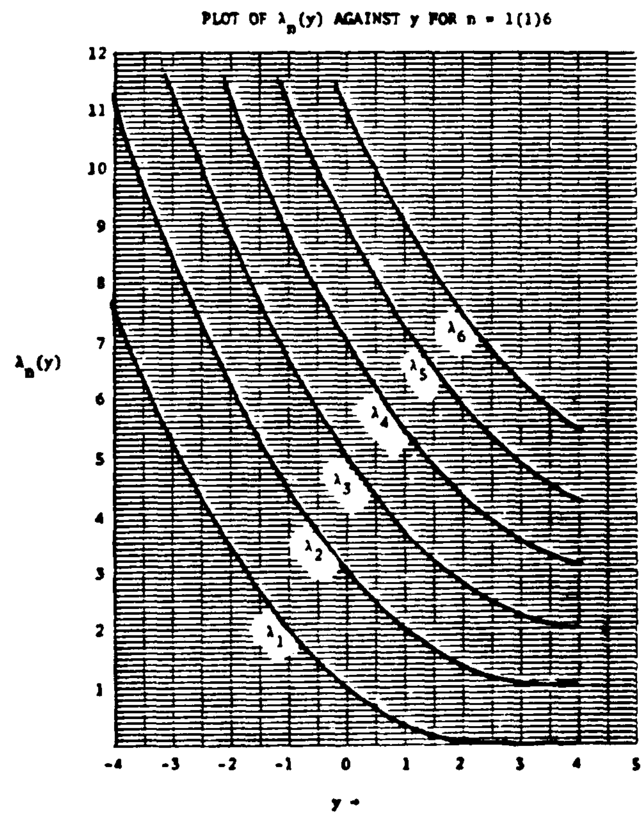


FIGURE 2

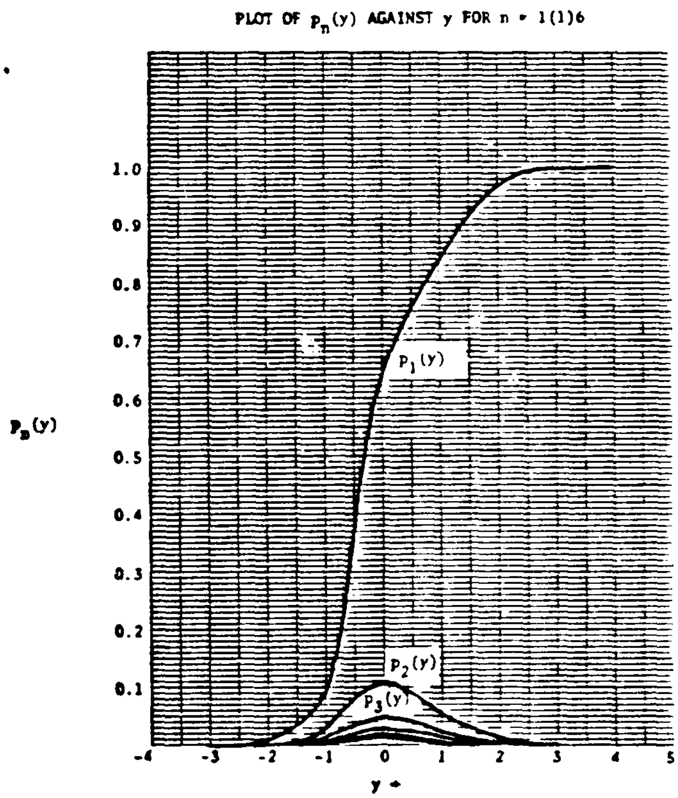


FIGURE 3

From (7), (8), (9a) and (10a) one obtains the asymptotic relation

$$(11) \qquad F_\theta(y) \sim p_1(y)F_0(y)e^{-\lambda_1(y)\theta} \quad , \quad y \to \infty \quad ,$$

and the corresponding approximation

$$(12) \qquad F_\theta(y) \approx p_1(y)F_0(y)e^{-\lambda_1(y)\theta} \quad .$$

It is found that

$$(13) \qquad 0 \le F_\theta(y) - p_1(y)F_0(y)e^{-\lambda_1(y)\theta} \le 10^{-4} \quad , \quad \theta > 12 \ ; \quad -\infty < y < \infty \quad ,$$

i.e., we have accuracy for (12) to four significant figures for all real y when $\theta > 12$ or, say, $c(\theta) \ge 2.5$.

It may be noted that since (6) is true for all θ, y, we may replace θ by $\mu_{0y}\tau$ where $\mu_{0y} = E[T_y^E]$. Hence

$$(14) \qquad P[M(\mu_{0y}\tau) < y] = F_0(y)P[T_y^E/\mu_{0y} > \tau] \quad .$$

But $E[T_y^E] \sim E[T_{0y}]$, $y \to \infty$ and $P[(T_y^E/\mu_{Ey}) > \tau] \to e^{-\tau}$, as $\tau \to \infty$ [4, 11]. Hence

$$(15) \qquad \lim_{y\to\infty} P[M(\tau) < \mu_{0y}^{INV}(y/\tau)] = e^{-\tau}$$

where $\mu_{0y}^{INV}(x)$ is the inverse function to $x = \mu_{0y}$, which is increasing on

$(0,\infty)$ with y. It is this relationship which is the basis of the extreme value limit theorem (see Appendix 3), and in that sense the limit theorem is equivalent to the asymptotic relation (11).

Acknowledgment

The authors wish to thank I. I. Gringorten for initiating this study and for many helpful discussions. We also wish to thank J. A. Wellner for valuable comments and suggestions, and U. Sumita and Ms. L. Ziegenfuss for technical and editorial contributions.

REFERENCES

[1] Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models, Holt, Rinehart and Winston, New York.

[2] Cramer, H. (1951). Mathematical Methods of Statistics, Princeton University Press, Princeton, NJ.

[3] Cramer, H. and Leadbetter, M. R. (1967). Stationary and Related Stochastic Processes, John Wiley, New York.

[4] Darling, D. A. and Erdos, P. (1955). "A limit theorem for the maximum of normalized sums of independent random variables," Ann. Math. Stat., 26, pp. 294-300.

[5] Darling, D. A. and Siegert, A.J.F. (1953). "The first passage problem for a continuous Markov process," Ann. Math. Stat., 24, pp. 624-639.

[6] Gnedenko, B. V. (1943). "Sur la distribution limite du terme maximum d'une série aléatoire," Ann. of Math., 44, pp. 423-453.

[7] Gumbel, E. J. (1958). Statistics of Extremes, New York: Columbia University Press.

[8] Gringorten, I. I. (1966). "A stochastic model of the frequency and duration of weather events," J. App. Meteorology, 5, pp. 606-624.

[9] Gringorten, I. I. (1968). "Estimating finite-time maxima and minima of a stationary Gaussian Ornstein-Uhlenbeck process by Monte Carlo simulation," Amer. Stat. Assn. J., pp. 1517-21.

[10] Keilson, J. and Ross, H. F. (1975). "Passage time distributions for Gaussian Markov statistical processes," Selected Tables in Mathematical Statistics, Vol. III (edited by the Institute of Mathematical Statistics). American Math. Society, Providence, Rhode Island.

[11] Keilson, J. (1974). "Sojourn times, exit times and jitter in multivariate Markov processes," Adv. Appl. Prob., 6, pp. 747-756.

[12] Magnus, W, Oberhettinger, F. and Soni, R. P. (1966). Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag, New York.

Appendix 1

The function $D_w(z)$ and its differentiation with respect to w

The parabolic cylinder function $D_w(z)$ is related to the Gamma function and confluent hypergeometric function by [12], p. 324)

$$(1) \qquad D_w(z) = 2^{w/2} e^{-z^2/4} \left\{ \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1-w}{2})} \, {}_1F_1(-\frac{w}{2}; \frac{1}{2}; \frac{z^2}{2}) \right.$$

$$\left. + \, 2^{-1/2} z \, \frac{\Gamma(-1/2)}{\Gamma(-w/2)} \, {}_1F_1(\frac{1-w}{2}; \frac{3}{2}; \frac{z^2}{2}) \right\} ,$$

where

$$(2) \qquad {}_1F_1(v;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(v)} \sum_{n=0}^{\infty} \frac{\Gamma(v+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

and this representation is valid for all complex w and z. The factor $2^{w/2}\exp(-z^2/4)$ is entire in w and z and $\Gamma^{-1}(v)$ is entire in v. Hence, any singularities of $D_w(z)$ in the complex w plane must be associated with ${}_1F_1(v;\ c;\ z)$. We note that

$$\left|\frac{\Gamma(v+n)}{\Gamma(v)}\right| = |v(v+1)\ \dots\ (v+n-1)| \le |v|(|v|+1)\ \dots\ (|v|+n-1)$$

so that $|{}_1F_1(v;\ c;\ z)| \le |{}_1F_1(|v|;\ c;\ z)|$. Since $v(v+1)\dots(v+n-1)$ is entire in v, it follows from the inequality above and the Weierstrass M-test that ${}_1F_1(v;\ c;\ z)$ is entire in v and z and $D_w(z)$ is entire in w and z. As

shown in [11], $D_{-s}(-y)$ has zeros only for s real and negative.

To derive equation (2.10), we note that

$$(3) \qquad \int_{-\infty}^{y} e^{-x^2/4} D_{-s}(-x)dx = \int_{-y}^{\infty} e^{-x^2/4} D_{-s}(x)dx$$

$$= \int_{-y}^{\infty} - \frac{d}{dx}[e^{-x^2/4} D_{-s-1}(x)]dx = e^{-y^2/4} D_{-s-1}(-y) \quad .$$

The second equality is obtained from [12, p. 327] with the misprint there corrected. Equation (2.10) then follows from (2.9), and the representation

$$(4) \qquad \frac{D_{-s-1}(-y)}{D_{-s}(-y)} = \sum_{n=1}^{\infty} \frac{\beta_n(y)}{s + \lambda_n(y)}$$

arising from the poles at $-\lambda_n(y)$ and the residues there. The positivity of $\beta_n(y)$ may be derived from the complete monotonicity of the ergodic exit time density $s_y^E(\tau)$ discussed in Section 2B. Equation (2.11) is obtained from (2.8) and (2.10) after observing from (2.10) that $\sigma_{xy}(0) = 1$ implies that

$$(5) \qquad \int_{-\infty}^{y} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\beta_n(y)}{\lambda_n(y)} \quad .$$

To verify equation (15) and thence (16), we proceed as follows. From (2.8) and (2.10) we have

$$(6) \qquad L_s\{F_\theta(y)\} = \int_{-\infty}^{y} \frac{e^{-x^2/2}}{s\sqrt{2\pi}} dx - \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \frac{D_{-s-1}(-y)}{sD_{-s}(-y)}$$

so that

$$(7) \qquad L_s\{f_\theta(y)\} = \frac{\partial}{\partial y} L_s\{F_\theta(y)\} = \frac{e^{-y^2/2}}{s\sqrt{2\pi}} - \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial y}\left[e^{-y^2/2} \frac{D_{-s-1}(-y)}{sD_{-s}(-y)}\right]$$

Also, from [12, pp. 326, 327]

$$(8) \qquad \frac{\partial}{\partial y}[e^{-y^2/4}D_{-s-1}(-y)] = e^{-y^2/4}D_{-s}(-y)$$

and

$$(9) \qquad \frac{\partial}{\partial y}[e^{y^2/4}D_{-s}(-y)] = se^{y^2/4}D_{-s-1}(-y) \quad .$$

Thus

$$\frac{\partial}{\partial y}\left[e^{-y^2/2} \frac{D_{-s-1}(-y)}{sD_{-s}(-y)}\right] = \frac{\partial}{\partial y}\left[\frac{e^{-y^2/4}D_{-s-1}(-y)}{se^{y^2/4}D_{-s}(-y)}\right]$$

$$= \frac{e^{-y^2/4}D_{-s}(-y)e^{+y^2/4}D_{-s}(-y) - se^{-y^2/4}D_{-s-1}(-y)e^{+y^2/4}D_{-s-1}(-y)}{se^{y^2/2}[D_{-s}(-y)]^2}$$

$$= \frac{e^{-y^2/2}}{s} - e^{-y^2/2}\left[\frac{D_{-s-1}(-y)}{D_{-s}(-y)}\right]^2 \quad .$$

It follows finally that

$$L_s\{f_\theta(y)\} = \frac{e^{-y^2/2}}{s\sqrt{2\pi}} - \frac{e^{-y^2/2}}{s\sqrt{2\pi}} + \frac{e^{-y^2/2}}{\sqrt{2\pi}}\left[\frac{D_{-s-1}(-y)}{D_{-s}(-y)}\right]^2 ,$$

i.e.,

$$(10) \qquad L_s\{f_\theta(y)\} = \frac{e^{-y^2/2}}{\sqrt{2\pi}}\left[\frac{D_{-s-1}(-y)}{D_{-s}(-y)}\right]^2$$

establishing (2.15).

Appendix 2

Calculations for $P[M(\theta) < 0] = F_\theta(0)$

The p.d.f. of T_{x0}, the passage time from $x < 0$ to $x = 0$ is known [11] to have the form

$$(1) \qquad s(x,0,\tau) = \frac{2x(1+u)\exp\{-\frac{1}{2}x^2/u\}}{\sqrt{2\pi}\,u^{3/2}}$$

where $u = e^{2\tau} - 1$. Hence the ergodic exit time density from $(-\infty,0)$ is

$$(2) \qquad s_E(\tau) = 2\int_{-\infty}^{0} \frac{e^{-x^2/2}}{\sqrt{2\pi}} s(x,0,\tau)dx$$

$$= \frac{2}{\pi} u^{-1/2} = \frac{2}{\pi} e^{-\tau}(1 - e^{-2\tau})^{-1/2} \quad ,$$

i.e., one has (2.19). Eq (2.20) then follows and the coefficients are found from the identity

$$(3) \qquad \frac{\frac{2}{\pi}w}{\sqrt{1 - w^2}} = \frac{2}{\pi} w\, F(\tfrac{1}{2}, b; b; w^2) \text{ , any } b \text{ ,}$$

$$= \frac{2}{\pi} \frac{w}{\Gamma(1/2)} \sum_{n=0}^{\infty} \Gamma(1/2 + n) \frac{w^{2n}}{n!} \quad ,$$

where F is the hypergeometric function. The asymptotic relation (2.22) is obtained from Stirling's Formula.

Appendix 3

Derivation of the limit theorem for extreme values

The limit theorem for $M(\theta)$ and $F_\theta(y)$ may be extracted from the contents of Darling and Erdos in the following way.

Def. 1. Let $X(t)$ be the stationary O-U process as described in Section 1, and let $M(t)$ be the maximum process as described there.

Def. 2. Let $T(\alpha)$ be the passage time of $X(t)$ to α, i.e., $T(\alpha) = \sup[t|x(\tau) < \alpha,\ 0 \le \tau \le t]$.

Def. 3. Let $t_k = \frac{1}{2} \log k$.

Def. 4. Let $U_n = \max\limits_{1\le k\le n} X(t_k) = M^*(t_n)$.

Def. 5. Let $N(\alpha)$, $1 \le k \le n$, be the first time that U_n reaches the set $[\alpha,\infty)$.

Then Darling and Erdos show that

(a) $$P[U_n < \alpha] = P[N(\alpha) > n] = P[\log N(\alpha) > \log n] \qquad \text{(L.3.2)}$$

(b) $$P[T(\alpha) > \sqrt{2\pi}\,\{e^{\alpha^2/2}/\alpha\}y] \to e^{-y}\ , \quad \alpha \to \infty \qquad \text{(L.3.4)}$$

(c) $$\log N(\alpha) - 2T(\alpha) \xrightarrow{P} 0\ , \quad \alpha \to \infty \qquad \text{(L.3.8)}$$

Hence, from Def. 4, and (a)

$$P[M^*(t_n) < \alpha] = P[\log N(\alpha) > \log n] \quad , \quad \text{all } \alpha, n \quad .$$

Let $n_{\alpha,y} \overset{\text{def}}{=} e^{2\mu(\alpha)y}$ where $\mu(\alpha) = \frac{\sqrt{2\pi}}{\alpha} e^{\alpha^2/2}$, so that

(1) $$P[M^*(\tfrac{1}{2}\log n_{\alpha,y}) < \alpha] = P[\log N(\alpha) > \log n_{\alpha,y}] \quad , \quad \text{all } \alpha \quad .$$

From (b), (c)

(2) $$\lim_{\alpha\to\infty} P[M^*(\tfrac{1}{2}\log n_{\alpha,y}) < \alpha] = \lim_{\alpha\to\infty} P[T(\alpha) > \mu(\alpha)y] = e^{-y} \quad \text{for all fixed } y > 0 \quad .$$

If we now set

(3) $$\frac{1}{2}\log n_{\alpha,y} = \mu(\alpha)y = \theta$$

and solve (3) for α as a function of y and θ, Eq. (2) becomes

(4) $$\lim_{\theta\to\infty} P[M^*(\theta) < \alpha(y,\theta)] = e^{-y} \quad , \quad \text{all } y > 0 \quad .$$

But (3) has the form $\sqrt{2\pi}\,\alpha^{-1} e^{\frac{1}{2}\alpha^2} y = \theta$, and its solution for α is given asymptotically (see below) by

(5) $$\alpha(y,\theta) \sim c(\theta) + \frac{\log c(\theta)}{c(\theta)} - \frac{1}{2}\,\frac{\log(2\pi y^2)}{c(\theta)} \quad , \quad \theta \to \infty$$

where

(6) $$c(\theta) = \sqrt{2\log\theta}$$

and this is the form of the limit theorem quoted in Section 3. Darling and Erdos state, without supplying details, that $M^*(\theta)$ and $M'(\theta)$ have the same asymptotic behavior. It is easier to verify this directly from (3.5), which states that

$$(7) \qquad P[M(\theta) < y] = P[M(0) < y]P[T_y^E > \theta] \quad .$$

Hence

$$(8) \qquad P[M(\mu_{0y}w) < y] = P[M(0) < y]P[(T_y^E/\mu_{0y}) > w] \quad .$$

But $E[T_y^E] \sim \mu_{0y} = E[T_{0y}]$ as $y \to \infty$ and $P[T_y^E/\mu_y^E > w] \to e^{-w}$, as $y \to \infty$. From (8) we then have

$$(9) \qquad \lim_{\theta\to\infty} P[M(\theta) < y(\theta,w)] = e^{-w}$$

where $y(\theta,w)$ is the solution of

$$(10) \qquad \theta = w\mu_{0y} \sim \sqrt{2\pi}\, \frac{e^{-y^2/2}w}{y} , \quad y \to \infty$$

so that (5) and the limit theorem follow.

<u>Asymptotic solution</u>

The equation to be solved may be written as $v = e^x/x$ where $v = \theta^2/(2\pi y^2)$, $x = \alpha^2$. Then $x = \log v + \log x \Rightarrow x = \log v + \log\log v + o(\log\log v)$, $v \to \infty$ and $\alpha = \sqrt{\log v + \log\log v} = w + (\log w)/w + o((\log w)/w)$ where

$w = \sqrt{\log v} = \sqrt{2\log\theta - \log(2\pi y^2)} = c(\theta) - \frac{1}{2}\log(2\pi y^2)/c(\theta)$. Hence $(\log w)/w = (\log c(\theta))/c(\theta) + o(c^{-1})$, and finally one obtains Eq. (5), and thence Eq. (4).

1

TABLE 1

C	0.0	0.0500	0.1000	0.1500	0.2000	0.2500
THETA	1.0000	1.0013	1.0050	1.0113	1.0202	1.0317
Y						
-4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-2.20	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-2.10	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001
-2.00	0.0003	0.0003	0.0002	0.0002	0.0002	0.0002
-1.90	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-1.80	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005
-1.70	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008
-1.60	0.0013	0.0013	0.0013	0.0012	0.0012	0.0012
-1.50	0.0019	0.0019	0.0019	0.0018	0.0018	0.0017
-1.40	0.0027	0.0027	0.0027	0.0026	0.0026	0.0025
-1.30	0.0038	0.0038	0.0038	0.0037	0.0036	0.0035
-1.20	0.0054	0.0053	0.0053	0.0052	0.0051	0.0050
-1.10	0.0074	0.0074	0.0073	0.0072	0.0071	0.0069
-1.00	0.0101	0.0101	0.0100	0.0099	0.0097	0.0095
-0.90	0.0137	0.0137	0.0136	0.0134	0.0132	0.0129
-0.80	0.0183	0.0182	0.0181	0.0179	0.0176	0.0172
-0.70	0.0241	0.0240	0.0239	0.0236	0.0233	0.0228
-0.60	0.0314	0.0313	0.0311	0.0308	0.0304	0.0298
-0.50	0.0404	0.0403	0.0401	0.0397	0.0392	0.0385
-0.40	0.0513	0.0513	0.0510	0.0505	0.0499	0.0491
-0.30	0.0646	0.0644	0.0641	0.0636	0.0629	0.0619
-0.20	0.0802	0.0801	0.0798	0.0791	0.0783	0.0772
-0.10	0.0986	0.0985	0.0981	0.0974	0.0964	0.0952

2

TABLE 1

C	0.3000	0.3500	0.4000	0.4500	0.5000	0.5500
THETA	1.0460	1.0632	1.0833	1.1066	1.1331	1.1633
Y						
-4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
-2.20	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-2.10	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-2.00	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001
-1.90	0.0003	0.0003	0.0003	0.0003	0.0002	0.0002
-1.80	0.0005	0.0005	0.0004	0.0004	0.0004	0.0003
-1.70	0.0008	0.0007	0.0007	0.0006	0.0006	0.0005
-1.60	0.0011	0.0011	0.0010	0.0010	0.0009	0.0008
-1.50	0.0017	0.0016	0.0015	0.0014	0.0013	0.0012
-1.40	0.0024	0.0023	0.0022	0.0021	0.0019	0.0018
-1.30	0.0034	0.0033	0.0031	0.0030	0.0028	0.0026
-1.20	0.0048	0.0046	0.0044	0.0042	0.0039	0.0037
-1.10	0.0067	0.0065	0.0062	0.0059	0.0056	0.0052
-1.00	0.0092	0.0089	0.0086	0.0082	0.0077	0.0073
-0.90	0.0125	0.0121	0.0117	0.0111	0.0106	0.0100
-0.80	0.0168	0.0163	0.0157	0.0151	0.0143	0.0136
-0.70	0.0223	0.0216	0.0209	0.0201	0.0192	0.0182
-0.60	0.0291	0.0283	0.0274	0.0264	0.0253	0.0242
-0.50	0.0377	0.0367	0.0356	0.0344	0.0331	0.0316
-0.40	0.0481	0.0470	0.0457	0.0442	0.0426	0.0409
-0.30	0.0608	0.0594	0.0579	0.0562	0.0543	0.0522
-0.20	0.0759	0.0743	0.0725	0.0705	0.0683	0.0658
-0.10	0.0936	0.0918	0.0898	0.0875	0.0849	0.0821

3

TABLE 1

C	0.6000	0.6500	0.7000	0.7500	0.8000	0.8500
THETA	1.1972	1.2352	1.2776	1.3248	1.3771	1.4351
Y						
-4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.10	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000
-2.00	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-1.90	0.0002	0.0002	0.0002	0.0001	0.0001	0.0001
-1.80	0.0003	0.0003	0.0002	0.0002	0.0002	0.0001
-1.70	0.0005	0.0004	0.0004	0.0003	0.0003	0.0002
-1.60	0.0007	0.0007	0.0006	0.0005	0.0004	0.0004
-1.50	0.0011	0.0010	0.0009	0.0008	0.0007	0.0006
-1.40	0.0016	0.0015	0.0013	0.0012	0.0010	0.0009
-1.30	0.0024	0.0022	0.0020	0.0017	0.0015	0.0013
-1.20	0.0034	0.0031	0.0028	0.0025	0.0023	0.0020
-1.10	0.0048	0.0045	0.0041	0.0037	0.0033	0.0029
-1.00	0.0068	0.0063	0.0058	0.0052	0.0047	0.0042
-0.90	0.0094	0.0087	0.0080	0.0073	0.0066	0.0059
-0.80	0.0128	0.0119	0.0111	0.0102	0.0092	0.0083
-0.70	0.0172	0.0161	0.0150	0.0139	0.0127	0.0115
-0.60	0.0229	0.0215	0.0201	0.0187	0.0172	0.0157
-0.50	0.0301	0.0284	0.0267	0.0249	0.0230	0.0211
-0.40	0.0390	0.0370	0.0349	0.0327	0.0304	0.0281
-0.30	0.0499	0.0475	0.0450	0.0424	0.0396	0.0368
-0.20	0.0632	0.0604	0.0574	0.0542	0.0510	0.0476
-0.10	0.0790	0.0758	0.0723	0.0686	0.0647	0.0607

4

TABLE 1

C	0.9000	0.9500	1.0000	1.0500	1.1000	1.1500
THETA	1.4993	1.5703	1.6487	1.7354	1.8313	1.9372
Y						
-4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.90	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
-1.80	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000
-1.70	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001
-1.60	0.0003	0.0003	0.0002	0.0002	0.0001	0.0001
-1.50	0.0005	0.0004	0.0003	0.0003	0.0002	0.0002
-1.40	0.0008	0.0006	0.0005	0.0004	0.0003	0.0002
-1.30	0.0011	0.0010	0.0008	0.0007	0.0005	0.0004
-1.20	0.0017	0.0015	0.0012	0.0010	0.0008	0.0006
-1.10	0.0025	0.0022	0.0018	0.0015	0.0012	0.0010
-1.00	0.0037	0.0032	0.0027	0.0023	0.0019	0.0015
-0.90	0.0053	0.0046	0.0040	0.0034	0.0028	0.0023
-0.80	0.0074	0.0065	0.0057	0.0049	0.0041	0.0034
-0.70	0.0103	0.0092	0.0081	0.0070	0.0059	0.0050
-0.60	0.0142	0.0127	0.0112	0.0098	0.0085	0.0072
-0.50	0.0192	0.0173	0.0155	0.0136	0.0118	0.0102
-0.40	0.0257	0.0233	0.0210	0.0186	0.0164	0.0142
-0.30	0.0339	0.0310	0.0280	0.0251	0.0222	0.0194
-0.20	0.0441	0.0405	0.0369	0.0333	0.0298	0.0263
-0.10	0.0566	0.0523	0.0480	0.0437	0.0393	0.0351

5

TABLE 1

C	1.2000	1.2500	1.3000	1.3500	1.4000	1.4500
THETA	2.0544	2.1842	2.3280	2.4874	2.6645	2.8612
Y						
-4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.60	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
-1.50	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000
-1.40	0.0002	0.0001	0.0001	0.0001	0.0000	0.0000
-1.30	0.0003	0.0002	0.0002	0.0001	0.0001	0.0000
-1.20	0.0005	0.0004	0.0003	0.0002	0.0001	0.0001
-1.10	0.0008	0.0006	0.0004	0.0003	0.0002	0.0001
-1.00	0.0012	0.0009	0.0007	0.0005	0.0004	0.0002
-0.90	0.0018	0.0014	0.0011	0.0008	0.0006	0.0004
-0.80	0.0028	0.0022	0.0017	0.0013	0.0009	0.0007
-0.70	0.0041	0.0033	0.0026	0.0020	0.0015	0.0011
-0.60	0.0060	0.0049	0.0039	0.0030	0.0023	0.0017
-0.50	0.0086	0.0071	0.0058	0.0046	0.0035	0.0027
-0.40	0.0121	0.0101	0.0083	0.0067	0.0053	0.0041
-0.30	0.0168	0.0142	0.0119	0.0097	0.0078	0.0061
-0.20	0.0229	0.0197	0.0166	0.0138	0.0112	0.0089
-0.10	0.0309	0.0268	0.0229	0.0193	0.0159	0.0129

6

TABLE 1

C	1.5000	1.5500	1.6000	1.6500	1.7000	1.7500
THETA	3.0802	3.3243	3.5966	3.9011	4.2419	4.6240
Y						
-4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.10	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
-1.00	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
-0.90	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000
-0.80	0.0005	0.0003	0.0002	0.0001	0.0001	0.0000
-0.70	0.0007	0.0005	0.0003	0.0002	0.0001	0.0001
-0.60	0.0012	0.0008	0.0005	0.0003	0.0002	0.0001
-0.50	0.0019	0.0014	0.0009	0.0006	0.0004	0.0002
-0.40	0.0030	0.0022	0.0015	0.0010	0.0006	0.0004
-0.30	0.0046	0.0034	0.0024	0.0016	0.0011	0.0007
-0.20	0.0069	0.0052	0.0038	0.0026	0.0018	0.0011
-0.10	0.0101	0.0078	0.0058	0.0042	0.0029	0.0019

7

TABLE 1

C	1.8000	1.8500	1.9000	1.9500	2.0000	2.0500
THETA	5.0531	5.5359	6.0800	6.6943	7.3891	8.1764
Y						
-4.00	0.0000	0.0000	0.0000	0.0	0.0	0.0
-3.90	0.0000	0.0000	0.0000	0.0000	0.0	0.0
-3.80	0.0000	0.0000	0.0000	0.0000	0.0	0.0
-3.70	0.0000	0.0000	0.0000	0.0000	0.0	0.0
-3.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
-3.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
-3.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
-3.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.60	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
-0.50	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
-0.40	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
-0.30	0.0004	0.0002	0.0001	0.0000	0.0000	0.0000
-0.20	0.0007	0.0004	0.0002	0.0001	0.0000	0.0000
-0.10	0.0012	0.0007	0.0004	0.0002	0.0001	0.0000

8

TABLE 1

C	2.1000	2.1500	2.2000	2.2500	2.3000	2.3500
THETA	9.0703	10.0870	11.2459	12.5692	14.0834	15.8196
Y						
-4.00	0.0	0.0	0.0	0.0	0.0	0.0
-3.90	0.0	0.0	0.0	0.0	0.0	0.0
-3.80	0.0	0.0	0.0	0.0	0.0	0.0
-3.70	0.0	0.0	0.0	0.0	0.0	0.0
-3.60	0.0	0.0	0.0	0.0	0.0	0.0
-3.50	0.0	0.0	0.0	0.0	0.0	0.0
-3.40	0.0	0.0	0.0	0.0	0.0	0.0
-3.30	0.0	0.0	0.0	0.0	0.0	0.0
-3.20	0.0	0.0	0.0	0.0	0.0	0.0
-3.10	0.0000	0.0	0.0	0.0	0.0	0.0
-3.00	0.0000	0.0	0.0	0.0	0.0	0.0
-2.90	0.0000	0.0	0.0	0.0	0.0	0.0
-2.80	0.0000	0.0000	0.0	0.0	0.0	0.0
-2.70	0.0000	0.0000	0.0	0.0	0.0	0.0
-2.60	0.0000	0.0000	0.0	0.0	0.0	0.0
-2.50	0.0000	0.0000	0.0000	0.0	0.0	0.0
-2.40	0.0000	0.0000	0.0000	0.0	0.0	0.0
-2.30	0.0000	0.0000	0.0000	0.0000	0.0	0.0
-2.20	0.0000	0.0000	0.0000	0.0000	0.0	0.0
-2.10	0.0000	0.0000	0.0000	0.0000	0.0	0.0
-2.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
-1.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0
-1.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.80	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.50	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.40	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.30	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-0.10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

9

TABLE 1

C	0.0	0.0500	0.1000	0.1500	0.2000	0.2500
THETA	1.0000	1.0013	1.0050	1.0113	1.0202	1.0317
Y						
0.0	0.1199	0.1198	0.1193	0.1185	0.1174	0.1160
0.10	0.1442	0.1440	0.1435	0.1426	0.1414	0.1398
0.20	0.1716	0.1714	0.1708	0.1698	0.1685	0.1668
0.30	0.2020	0.2018	0.2012	0.2001	0.1987	0.1968
0.40	0.2354	0.2352	0.2345	0.2334	0.2319	0.2299
0.50	0.2716	0.2714	0.2707	0.2695	0.2679	0.2658
0.60	0.3104	0.3101	0.3094	0.3082	0.3065	0.3043
0.70	0.3513	0.3510	0.3503	0.3490	0.3472	0.3450
0.80	0.3939	0.3936	0.3929	0.3916	0.3898	0.3875
0.90	0.4378	0.4375	0.4368	0.4355	0.4337	0.4314
1.00	0.4824	0.4821	0.4814	0.4801	0.4784	0.4761
1.10	0.5271	0.5269	0.5262	0.5249	0.5232	0.5210
1.20	0.5715	0.5713	0.5706	0.5694	0.5677	0.5656
1.30	0.6149	0.6147	0.6140	0.6129	0.6113	0.6093
1.40	0.6569	0.6567	0.6560	0.6550	0.6535	0.6516
1.50	0.6969	0.6967	0.6961	0.6952	0.6938	0.6920
1.60	0.7347	0.7345	0.7340	0.7331	0.7318	0.7302
1.70	0.7698	0.7697	0.7692	0.7684	0.7672	0.7658
1.80	0.8022	0.8020	0.8016	0.8009	0.7999	0.7985
1.90	0.8316	0.8315	0.8311	0.8304	0.8295	0.8284
2.00	0.8580	0.8579	0.8575	0.8570	0.8562	0.8552
2.10	0.8814	0.8813	0.8810	0.8805	0.8798	0.8789
2.20	0.9019	0.9018	0.9015	0.9011	0.9006	0.8998
2.30	0.9196	0.9196	0.9194	0.9190	0.9185	0.9179
2.40	0.9348	0.9348	0.9346	0.9343	0.9339	0.9334
2.50	0.9477	0.9476	0.9475	0.9473	0.9469	0.9465
2.60	0.9584	0.9584	0.9583	0.9581	0.9578	0.9574
2.70	0.9673	0.9672	0.9672	0.9670	0.9668	0.9665
2.80	0.9745	0.9745	0.9744	0.9743	0.9741	0.9739
2.90	0.9803	0.9803	0.9803	0.9802	0.9800	0.9799
3.00	0.9850	0.9850	0.9849	0.9849	0.9848	0.9846
3.10	0.9887	0.9887	0.9886	0.9886	0.9885	0.9884
3.20	0.9915	0.9915	0.9915	0.9914	0.9914	0.9913
3.30	0.9937	0.9937	0.9937	0.9937	0.9936	0.9936
3.40	0.9954	0.9954	0.9954	0.9954	0.9953	0.9953
3.50	0.9967	0.9967	0.9967	0.9966	0.9966	0.9966
3.60	0.9976	0.9976	0.9976	0.9976	0.9976	0.9975
3.70	0.9983	0.9983	0.9983	0.9983	0.9983	0.9983
3.80	0.9988	0.9988	0.9988	0.9988	0.9988	0.9988
3.90	0.9992	0.9992	0.9992	0.9992	0.9992	0.9991
4.00	0.9994	0.9994	0.9994	0.9994	0.9994	0.9994
4.10	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
4.20	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
4.30	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
4.40	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
4.50	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

10

TABLE 1

C	0.3000	0.3500	0.4000	0.4500	0.5000	0.5500
THETA	1.0460	1.0632	1.0833	1.1066	1.1331	1.1633
Y						
0.0	0.1143	0.1122	0.1099	0.1073	0.1044	0.1012
0.10	0.1379	0.1357	0.1331	0.1301	0.1268	0.1232
0.20	0.1647	0.1622	0.1593	0.1561	0.1524	0.1484
0.30	0.1945	0.1918	0.1887	0.1852	0.1812	0.1768
0.40	0.2274	0.2245	0.2212	0.2174	0.2131	0.2084
0.50	0.2632	0.2601	0.2566	0.2525	0.2480	0.2430
0.60	0.3016	0.2984	0.2947	0.2904	0.2857	0.2804
0.70	0.3422	0.3389	0.3351	0.3307	0.3258	0.3204
0.80	0.3847	0.3814	0.3775	0.3730	0.3681	0.3625
0.90	0.4286	0.4252	0.4213	0.4169	0.4118	0.4062
1.00	0.4733	0.4700	0.4661	0.4617	0.4567	0.4511
1.10	0.5183	0.5150	0.5112	0.5069	0.5020	0.4965
1.20	0.5630	0.5598	0.5562	0.5520	0.5472	0.5419
1.30	0.6068	0.6038	0.6003	0.5963	0.5918	0.5867
1.40	0.6492	0.6464	0.6431	0.6394	0.6351	0.6303
1.50	0.6898	0.6872	0.6842	0.6806	0.6767	0.6722
1.60	0.7282	0.7258	0.7230	0.7197	0.7161	0.7120
1.70	0.7640	0.7618	0.7592	0.7563	0.7530	0.7493
1.80	0.7969	0.7950	0.7927	0.7901	0.7871	0.7838
1.90	0.8269	0.8252	0.8232	0.8209	0.8182	0.8153
2.00	0.8539	0.8524	0.8506	0.8486	0.8463	0.8437
2.10	0.8779	0.8766	0.8751	0.8733	0.8713	0.8691
2.20	0.8989	0.8978	0.8965	0.8950	0.8933	0.8914
2.30	0.9171	0.9162	0.9151	0.9138	0.9124	0.9108
2.40	0.9327	0.9320	0.9311	0.9300	0.9288	0.9275
2.50	0.9460	0.9453	0.9446	0.9437	0.9427	0.9416
2.60	0.9570	0.9565	0.9559	0.9552	0.9544	0.9535
2.70	0.9662	0.9657	0.9653	0.9647	0.9640	0.9633
2.80	0.9736	0.9733	0.9729	0.9725	0.9719	0.9714
2.90	0.9796	0.9794	0.9791	0.9787	0.9783	0.9779
3.00	0.9845	0.9843	0.9840	0.9838	0.9834	0.9831
3.10	0.9883	0.9881	0.9879	0.9877	0.9875	0.9872
3.20	0.9912	0.9911	0.9910	0.9908	0.9906	0.9904
3.30	0.9935	0.9934	0.9933	0.9932	0.9930	0.9929
3.40	0.9952	0.9952	0.9951	0.9950	0.9949	0.9948
3.50	0.9965	0.9965	0.9964	0.9964	0.9963	0.9962
3.60	0.9975	0.9975	0.9974	0.9974	0.9973	0.9973
3.70	0.9982	0.9982	0.9982	0.9981	0.9981	0.9981
3.80	0.9988	0.9987	0.9987	0.9987	0.9987	0.9986
3.90	0.9991	0.9991	0.9991	0.9991	0.9991	0.9991
4.00	0.9994	0.9994	0.9994	0.9994	0.9994	0.9993
4.10	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996
4.20	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
4.30	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
4.40	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
4.50	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

11

TABLE 1

C	0.6000	0.6500	0.7000	0.7500	0.8000	0.8500
THETA	1.1972	1.2352	1.2776	1.3248	1.3771	1.4351
Y						
0.0	0.0977	0.0939	0.0899	0.0857	0.0812	0.0765
0.10	0.1193	0.1151	0.1105	0.1057	0.1006	0.0952
0.20	0.1441	0.1394	0.1343	0.1289	0.1231	0.1171
0.30	0.1721	0.1669	0.1613	0.1553	0.1489	0.1422
0.40	0.2032	0.1976	0.1915	0.1850	0.1781	0.1707
0.50	0.2375	0.2315	0.2250	0.2180	0.2105	0.2025
0.60	0.2747	0.2683	0.2615	0.2540	0.2461	0.2376
0.70	0.3144	0.3078	0.3007	0.2929	0.2846	0.2757
0.80	0.3563	0.3496	0.3422	0.3343	0.3257	0.3164
0.90	0.4000	0.3932	0.3857	0.3777	0.3689	0.3594
1.00	0.4449	0.4381	0.4307	0.4225	0.4137	0.4042
1.10	0.4904	0.4837	0.4764	0.4684	0.4597	0.4502
1.20	0.5360	0.5295	0.5224	0.5145	0.5060	0.4968
1.30	0.5810	0.5748	0.5679	0.5604	0.5522	0.5433
1.40	0.6250	0.6190	0.6125	0.6054	0.5976	0.5891
1.50	0.6672	0.6617	0.6556	0.6489	0.6416	0.6336
1.60	0.7074	0.7023	0.6967	0.6905	0.6837	0.6763
1.70	0.7451	0.7405	0.7353	0.7297	0.7235	0.7167
1.80	0.7800	0.7758	0.7712	0.7662	0.7606	0.7544
1.90	0.8119	0.8083	0.8042	0.7996	0.7947	0.7892
2.00	0.8408	0.8376	0.8340	0.8300	0.8256	0.8208
2.10	0.8666	0.8637	0.8606	0.8572	0.8534	0.8492
2.20	0.8892	0.8868	0.8842	0.8812	0.8780	0.8744
2.30	0.9090	0.9070	0.9047	0.9022	0.8995	0.8965
2.40	0.9260	0.9243	0.9224	0.9203	0.9180	0.9155
2.50	0.9404	0.9390	0.9374	0.9357	0.9338	0.9317
2.60	0.9525	0.9513	0.9501	0.9487	0.9471	0.9454
2.70	0.9625	0.9616	0.9606	0.9595	0.9582	0.9568
2.80	0.9707	0.9700	0.9692	0.9683	0.9673	0.9662
2.90	0.9774	0.9768	0.9762	0.9755	0.9747	0.9738
3.00	0.9827	0.9823	0.9818	0.9812	0.9806	0.9799
3.10	0.9869	0.9866	0.9862	0.9858	0.9853	0.9848
3.20	0.9902	0.9899	0.9896	0.9893	0.9890	0.9886
3.30	0.9927	0.9925	0.9923	0.9921	0.9918	0.9915
3.40	0.9947	0.9945	0.9944	0.9942	0.9940	0.9938
3.50	0.9961	0.9960	0.9959	0.9958	0.9956	0.9955
3.60	0.9972	0.9971	0.9971	0.9970	0.9969	0.9967
3.70	0.9980	0.9980	0.9979	0.9978	0.9978	0.9977
3.80	0.9986	0.9986	0.9985	0.9985	0.9984	0.9984
3.90	0.9990	0.9990	0.9990	0.9989	0.9989	0.9989
4.00	0.9993	0.9993	0.9993	0.9993	0.9992	0.9992
4.10	0.9995	0.9995	0.9995	0.9995	0.9995	0.9995
4.20	0.9997	0.9997	0.9997	0.9997	0.9997	0.9996
4.30	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
4.40	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998
4.50	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

12

TABLE 1

C	0.9000	0.9500	1.0000	1.0500	1.1000	1.1500
THETA	1.4993	1.5703	1.6487	1.7354	1.8313	1.9372
Y						
0.0	0.0717	0.0667	0.0616	0.0564	0.0512	0.0460
0.10	0.0897	0.0839	0.0780	0.0719	0.0658	0.0596
0.20	0.1108	0.1042	0.0974	0.0904	0.0832	0.0760
0.30	0.1351	0.1277	0.1201	0.1121	0.1040	0.0957
0.40	0.1629	0.1547	0.1462	0.1373	0.1282	0.1188
0.50	0.1941	0.1852	0.1758	0.1660	0.1559	0.1454
0.60	0.2285	0.2190	0.2089	0.1983	0.1873	0.1758
0.70	0.2661	0.2560	0.2453	0.2340	0.2222	0.2098
0.80	0.3065	0.2960	0.2848	0.2729	0.2604	0.2472
0.90	0.3493	0.3385	0.3269	0.3146	0.3016	0.2879
1.00	0.3940	0.3830	0.3713	0.3588	0.3454	0.3313
1.10	0.4400	0.4291	0.4173	0.4047	0.3913	0.3770
1.20	0.4868	0.4760	0.4644	0.4519	0.4386	0.4243
1.30	0.5336	0.5231	0.5118	0.4997	0.4866	0.4726
1.40	0.5799	0.5698	0.5590	0.5473	0.5347	0.5211
1.50	0.6249	0.6155	0.6052	0.5941	0.5821	0.5692
1.60	0.6682	0.6594	0.6499	0.6395	0.6283	0.6161
1.70	0.7093	0.7012	0.6925	0.6829	0.6725	0.6612
1.80	0.7478	0.7404	0.7325	0.7238	0.7143	0.7040
1.90	0.7832	0.7767	0.7696	0.7618	0.7533	0.7440
2.00	0.8156	0.8098	0.8035	0.7966	0.7891	0.7808
2.10	0.8447	0.8396	0.8341	0.8281	0.8215	0.8143
2.20	0.8705	0.8662	0.8614	0.8563	0.8506	0.8443
2.30	0.8931	0.8895	0.8854	0.8810	0.8762	0.8709
2.40	0.9127	0.9096	0.9063	0.9026	0.8985	0.8940
2.50	0.9294	0.9269	0.9241	0.9210	0.9177	0.9140
2.60	0.9436	0.9415	0.9392	0.9367	0.9339	0.9309
2.70	0.9553	0.9536	0.9518	0.9498	0.9475	0.9451
2.80	0.9650	0.9637	0.9622	0.9606	0.9588	0.9568
2.90	0.9729	0.9718	0.9707	0.9694	0.9680	0.9664
3.00	0.9792	0.9784	0.9775	0.9765	0.9754	0.9742
3.10	0.9842	0.9836	0.9829	0.9821	0.9813	0.9804
3.20	0.9881	0.9877	0.9871	0.9866	0.9859	0.9852
3.30	0.9912	0.9908	0.9904	0.9900	0.9895	0.9890
3.40	0.9935	0.9933	0.9930	0.9926	0.9923	0.9919
3.50	0.9953	0.9951	0.9949	0.9946	0.9944	0.9941
3.60	0.9966	0.9965	0.9963	0.9961	0.9959	0.9957
3.70	0.9976	0.9975	0.9974	0.9972	0.9971	0.9970
3.80	0.9983	0.9982	0.9981	0.9981	0.9980	0.9979
3.90	0.9988	0.9988	0.9987	0.9986	0.9986	0.9985
4.00	0.9992	0.9991	0.9991	0.9991	0.9990	0.9990
4.10	0.9994	0.9994	0.9994	0.9994	0.9993	0.9993
4.20	0.9996	0.9996	0.9996	0.9996	0.9995	0.9995
4.30	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997
4.40	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
4.50	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

13

TABLE 1

C	1.2000	1.2500	1.3000	1.3500	1.4000	1.4500
THETA	2.0544	2.1842	2.3280	2.4874	2.6645	2.8612
Y						
0.0	0.0409	0.0359	0.0311	0.0265	0.0222	0.0182
0.10	0.0534	0.0474	0.0415	0.0358	0.0304	0.0253
0.20	0.0688	0.0616	0.0545	0.0476	0.0409	0.0346
0.30	0.0873	0.0788	0.0705	0.0622	0.0542	0.0465
0.40	0.1092	0.0995	0.0898	0.0801	0.0706	0.0614
0.50	0.1347	0.1237	0.1127	0.1015	0.0905	0.0796
0.60	0.1639	0.1518	0.1394	0.1268	0.1142	0.1017
0.70	0.1969	0.1836	0.1700	0.1560	0.1419	0.1277
0.80	0.2335	0.2192	0.2045	0.1893	0.1737	0.1579
0.90	0.2735	0.2584	0.2427	0.2264	0.2096	0.1924
1.00	0.3164	0.3007	0.2843	0.2671	0.2493	0.2309
1.10	0.3618	0.3458	0.3288	0.3111	0.2925	0.2732
1.20	0.4091	0.3929	0.3758	0.3577	0.3387	0.3187
1.30	0.4576	0.4416	0.4245	0.4064	0.3872	0.3670
1.40	0.5066	0.4909	0.4742	0.4564	0.4374	0.4172
1.50	0.5552	0.5402	0.5241	0.5068	0.4883	0.4686
1.60	0.6030	0.5888	0.5735	0.5570	0.5393	0.5203
1.70	0.6490	0.6358	0.6215	0.6060	0.5893	0.5713
1.80	0.6928	0.6807	0.6675	0.6532	0.6377	0.6210
1.90	0.7339	0.7229	0.7110	0.6979	0.6838	0.6684
2.00	0.7719	0.7621	0.7514	0.7397	0.7270	0.7131
2.10	0.8065	0.7978	0.7884	0.7781	0.7669	0.7545
2.20	0.8375	0.8301	0.8219	0.8129	0.8031	0.7923
2.30	0.8651	0.8587	0.8517	0.8440	0.8355	0.8262
2.40	0.8891	0.8837	0.8778	0.8713	0.8642	0.8563
2.50	0.9099	0.9054	0.9005	0.8950	0.8891	0.8825
2.60	0.9276	0.9239	0.9198	0.9153	0.9104	0.9050
2.70	0.9424	0.9394	0.9361	0.9325	0.9285	0.9240
2.80	0.9547	0.9523	0.9496	0.9467	0.9435	0.9400
2.90	0.9647	0.9628	0.9608	0.9584	0.9559	0.9531
3.00	0.9728	0.9714	0.9697	0.9679	0.9659	0.9637
3.10	0.9793	0.9782	0.9769	0.9755	0.9740	0.9723
3.20	0.9844	0.9836	0.9826	0.9815	0.9804	0.9791
3.30	0.9884	0.9877	0.9870	0.9862	0.9853	0.9844
3.40	0.9914	0.9910	0.9904	0.9898	0.9892	0.9884
3.50	0.9938	0.9934	0.9930	0.9926	0.9921	0.9916
3.60	0.9955	0.9952	0.9950	0.9946	0.9943	0.9939
3.70	0.9968	0.9966	0.9964	0.9962	0.9959	0.9956
3.80	0.9977	0.9976	0.9975	0.9973	0.9971	0.9969
3.90	0.9984	0.9983	0.9982	0.9981	0.9980	0.9978
4.00	0.9989	0.9988	0.9988	0.9987	0.9986	0.9985
4.10	0.9993	0.9992	0.9992	0.9991	0.9990	0.9990
4.20	0.9995	0.9995	0.9994	0.9994	0.9994	0.9993
4.30	0.9997	0.9996	0.9996	0.9996	0.9996	0.9995
4.40	0.9998	0.9998	0.9997	0.9997	0.9997	0.9997
4.50	0.9999	0.9998	0.9998	0.9998	0.9998	0.9998

14

TABLE 1

C	1.5000	1.5500	1.6000	1.6500	1.7000	1.7500
THETA	3.0802	3.3243	3.5966	3.9011	4.2419	4.6240
Y						
0.0	0.0146	0.0115	0.0087	0.0064	0.0046	0.0031
0.10	0.0207	0.0165	0.0129	0.0097	0.0071	0.0050
0.20	0.0287	0.0234	0.0185	0.0143	0.0107	0.0078
0.30	0.0392	0.0324	0.0262	0.0207	0.0159	0.0118
0.40	0.0525	0.0441	0.0363	0.0293	0.0230	0.0175
0.50	0.0691	0.0589	0.0494	0.0405	0.0325	0.0254
0.60	0.0893	0.0773	0.0659	0.0550	0.0450	0.0359
0.70	0.1136	0.0997	0.0862	0.0732	0.0610	0.0498
0.80	0.1421	0.1263	0.1107	0.0956	0.0811	0.0675
0.90	0.1749	0.1573	0.1398	0.1225	0.1057	0.0895
1.00	0.2120	0.1928	0.1735	0.1541	0.1350	0.1164
1.10	0.2532	0.2326	0.2117	0.1905	0.1693	0.1483
1.20	0.2979	0.2764	0.2541	0.2314	0.2084	0.1853
1.30	0.3458	0.3235	0.3004	0.2766	0.2521	0.2273
1.40	0.3959	0.3735	0.3499	0.3254	0.2999	0.2738
1.50	0.4476	0.4253	0.4018	0.3770	0.3511	0.3242
1.60	0.4999	0.4782	0.4551	0.4306	0.4048	0.3777
1.70	0.5520	0.5312	0.5089	0.4852	0.4599	0.4332
1.80	0.6028	0.5833	0.5623	0.5397	0.5154	0.4896
1.90	0.6518	0.6337	0.6142	0.5930	0.5702	0.5457
2.00	0.6980	0.6816	0.6638	0.6444	0.6233	0.6006
2.10	0.7411	0.7264	0.7104	0.6929	0.6738	0.6531
2.20	0.7805	0.7676	0.7534	0.7379	0.7209	0.7024
2.30	0.8161	0.8049	0.7925	0.7790	0.7642	0.7478
2.40	0.8476	0.8380	0.8275	0.8159	0.8031	0.7890
2.50	0.8752	0.8671	0.8583	0.8485	0.8376	0.8257
2.60	0.8989	0.8923	0.8849	0.8768	0.8677	0.8577
2.70	0.9191	0.9137	0.9077	0.9010	0.8936	0.8853
2.80	0.9360	0.9316	0.9268	0.9214	0.9154	0.9087
2.90	0.9499	0.9465	0.9426	0.9383	0.9335	0.9282
3.00	0.9613	0.9586	0.9555	0.9522	0.9484	0.9442
3.10	0.9704	0.9683	0.9659	0.9633	0.9604	0.9572
3.20	0.9776	0.9760	0.9742	0.9722	0.9700	0.9675
3.30	0.9833	0.9821	0.9807	0.9792	0.9775	0.9756
3.40	0.9876	0.9867	0.9857	0.9846	0.9833	0.9819
3.50	0.9910	0.9903	0.9895	0.9887	0.9878	0.9868
3.60	0.9935	0.9930	0.9924	0.9918	0.9912	0.9904
3.70	0.9953	0.9950	0.9946	0.9942	0.9937	0.9931
3.80	0.9967	0.9964	0.9962	0.9959	0.9955	0.9951
3.90	0.9977	0.9975	0.9973	0.9971	0.9969	0.9966
4.00	0.9984	0.9983	0.9981	0.9980	0.9978	0.9976
4.10	0.9989	0.9988	0.9987	0.9986	0.9985	0.9984
4.20	0.9993	0.9992	0.9991	0.9991	0.9990	0.9989
4.30	0.9995	0.9995	0.9994	0.9994	0.9993	0.9993
4.40	0.9997	0.9996	0.9996	0.9996	0.9995	0.9995
4.50	0.9998	0.9998	0.9997	0.9997	0.9997	0.9997

35

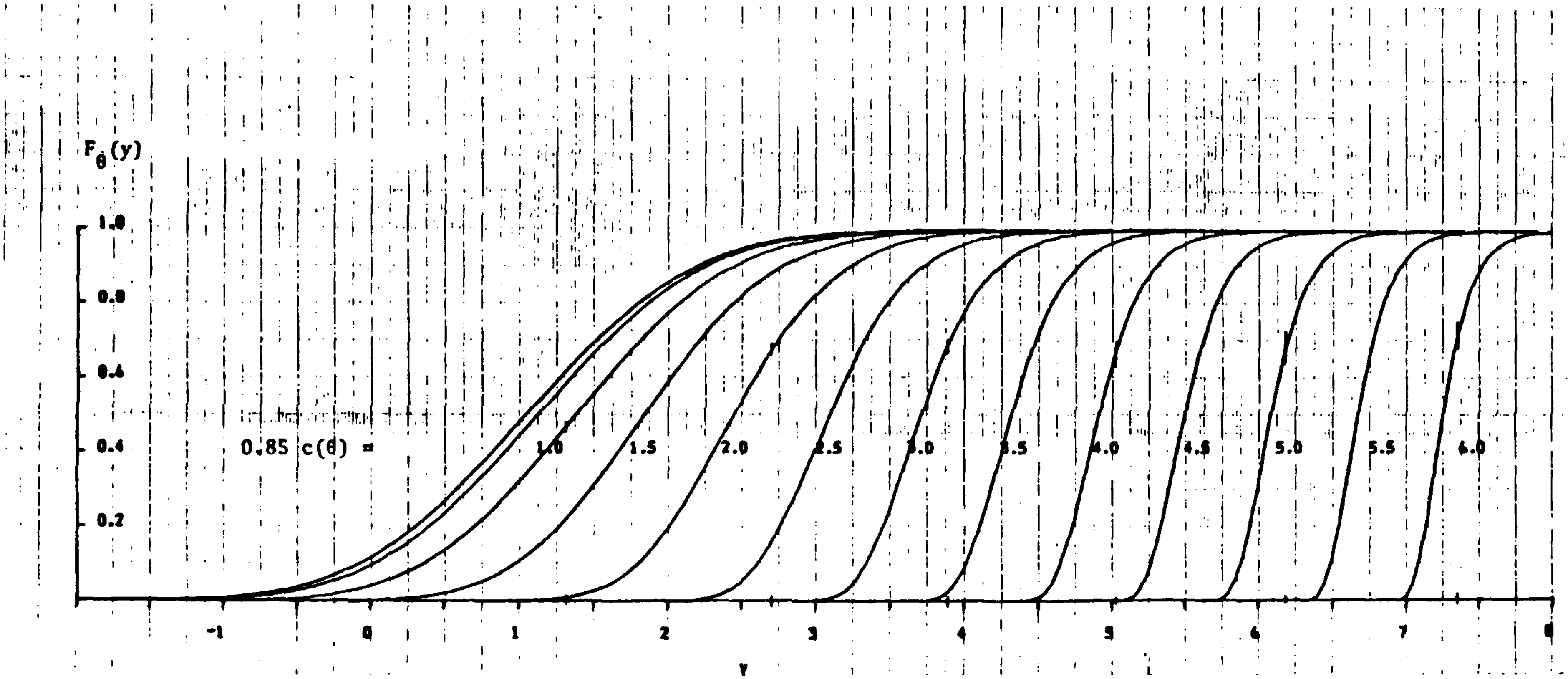
TABLE 1

C	1.8000	1.8500	1.9000	1.9500	2.0000	2.0500
THETA	5.0531	5.5359	6.0800	6.6943	7.3891	8.1764
V						
0.0	0.0020	0.0013	0.0007	0.0004	0.0002	0.0001
0.10	0.0034	0.0021	0.0013	0.0007	0.0004	0.0002
0.20	0.0054	0.0036	0.0023	0.0013	0.0007	0.0004
0.30	0.0084	0.0058	0.0038	0.0024	0.0014	0.0007
0.40	0.0129	0.0091	0.0062	0.0040	0.0024	0.0014
0.50	0.0192	0.0140	0.0099	0.0066	0.0042	0.0025
0.60	0.0279	0.0210	0.0152	0.0106	0.0070	0.0044
0.70	0.0396	0.0306	0.0229	0.0165	0.0114	0.0075
0.80	0.0549	0.0435	0.0334	0.0249	0.0178	0.0122
0.90	0.0743	0.0603	0.0476	0.0365	0.0270	0.0192
1.00	0.0985	0.0817	0.0661	0.0521	0.0398	0.0293
1.10	0.1278	0.1081	0.0896	0.0724	0.0569	0.0433
1.20	0.1624	0.1400	0.1184	0.0981	0.0792	0.0622
1.30	0.2023	0.1774	0.1531	0.1295	0.1073	0.0866
1.40	0.2471	0.2202	0.1934	0.1670	0.1415	0.1173
1.50	0.2964	0.2680	0.2393	0.2105	0.1821	0.1545
1.60	0.3494	0.3201	0.2900	0.2594	0.2287	0.1982
1.70	0.4050	0.3755	0.3448	0.3131	0.2808	0.2482
1.80	0.4621	0.4330	0.4024	0.3705	0.3374	0.3035
1.90	0.5195	0.4915	0.4617	0.4302	0.3972	0.3629
2.00	0.5760	0.5496	0.5212	0.4910	0.4589	0.4251
2.10	0.6305	0.6061	0.5797	0.5513	0.5209	0.4884
2.20	0.6821	0.6600	0.6359	0.6098	0.5816	0.5512
2.30	0.7299	0.7103	0.6888	0.6653	0.6397	0.6119
2.40	0.7735	0.7564	0.7375	0.7168	0.6941	0.6693
2.50	0.8124	0.7978	0.7816	0.7637	0.7440	0.7222
2.60	0.8466	0.8343	0.8206	0.8055	0.7886	0.7700
2.70	0.8762	0.8660	0.8546	0.8420	0.8279	0.8123
2.80	0.9013	0.8930	0.8837	0.8734	0.8618	0.8489
2.90	0.9222	0.9156	0.9082	0.8998	0.8905	0.8801
3.00	0.9395	0.9343	0.9284	0.9218	0.9144	0.9061
3.10	0.9535	0.9494	0.9448	0.9397	0.9339	0.9274
3.20	0.9647	0.9616	0.9580	0.9541	0.9496	0.9446
3.30	0.9735	0.9711	0.9685	0.9655	0.9621	0.9582
3.40	0.9803	0.9786	0.9766	0.9743	0.9718	0.9689
3.50	0.9856	0.9843	0.9828	0.9811	0.9793	0.9772
3.60	0.9896	0.9886	0.9875	0.9863	0.9849	0.9834
3.70	0.9925	0.9918	0.9911	0.9902	0.9892	0.9881
3.80	0.9947	0.9942	0.9937	0.9930	0.9923	0.9915
3.90	0.9963	0.9959	0.9955	0.9951	0.9946	0.9941
4.00	0.9974	0.9972	0.9969	0.9966	0.9963	0.9959
4.10	0.9982	0.9981	0.9979	0.9977	0.9974	0.9972
4.20	0.9988	0.9987	0.9986	0.9984	0.9983	0.9981
4.30	0.9992	0.9991	0.9990	0.9989	0.9988	0.9987
4.40	0.9995	0.9994	0.9994	0.9993	0.9992	0.9991
4.50	0.9997	0.9996	0.9996	0.9995	0.9995	0.9994

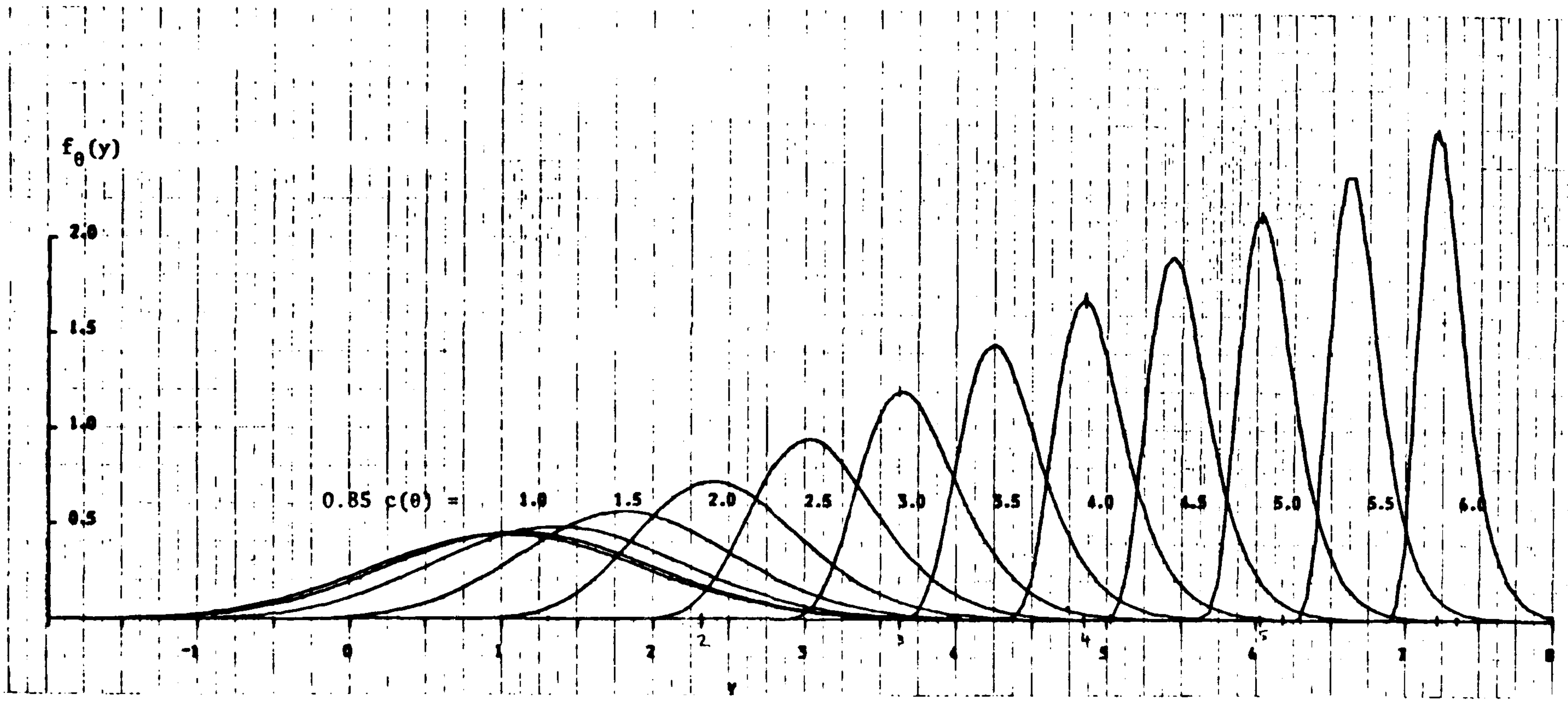
16

TABLE 1

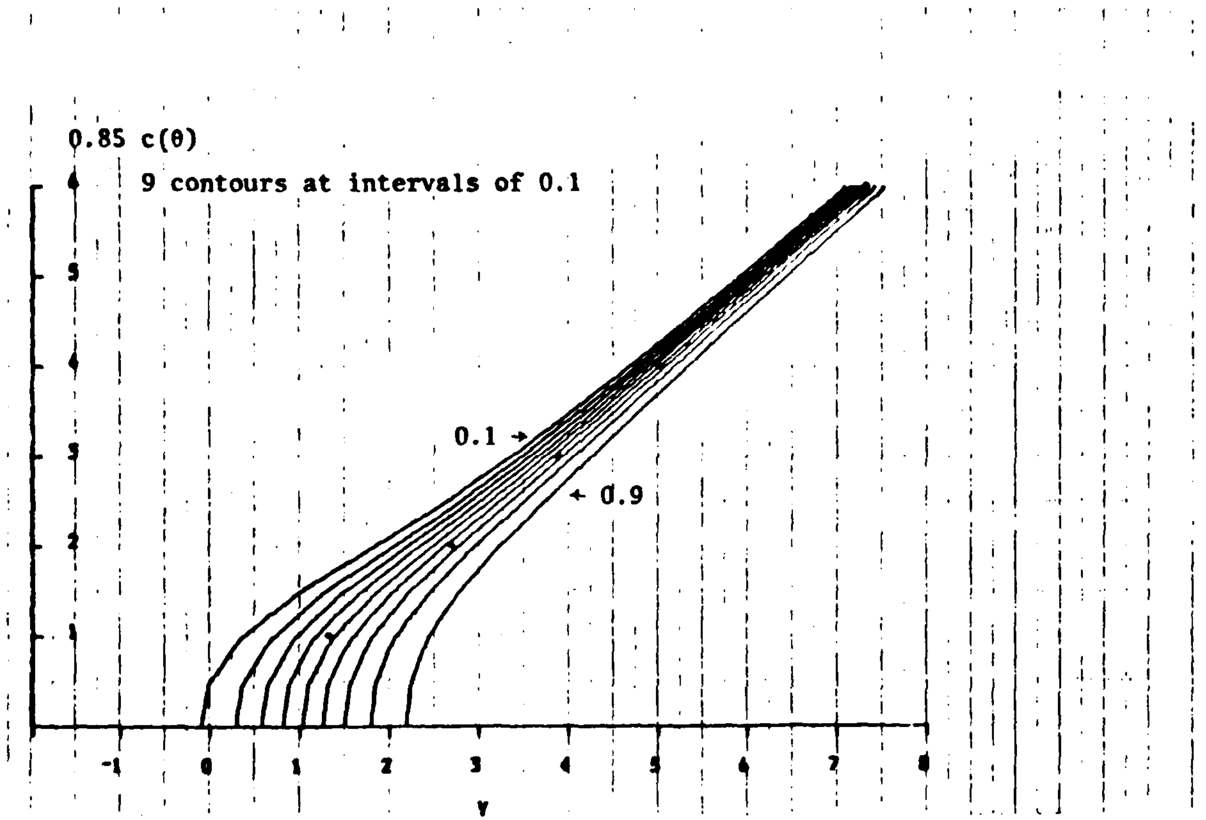
C	2.1000	2.1500	2.2000	2.2500	2.3000	2.3500
THETA	9.0703	10.0870	11.2459	12.5692	14.0834	15.8196
Y						
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.10	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.20	0.0002	0.0001	0.0000	0.0000	0.0000	0.0000
0.30	0.0004	0.0002	0.0001	0.0000	0.0000	0.0000
0.40	0.0007	0.0004	0.0002	0.0001	0.0000	0.0000
0.50	0.0014	0.0007	0.0003	0.0001	0.0001	0.0000
0.60	0.0026	0.0014	0.0007	0.0003	0.0001	0.0000
0.70	0.0046	0.0027	0.0014	0.0007	0.0003	0.0001
0.80	0.0079	0.0048	0.0028	0.0015	0.0007	0.0003
0.90	0.0130	0.0084	0.0051	0.0029	0.0015	0.0007
1.00	0.0207	0.0140	0.0089	0.0053	0.0030	0.0015
1.10	0.0318	0.0224	0.0150	0.0095	0.0056	0.0031
1.20	0.0473	0.0346	0.0242	0.0161	0.0101	0.0059
1.30	0.0680	0.0516	0.0376	0.0263	0.0174	0.0109
1.40	0.0947	0.0743	0.0564	0.0411	0.0286	0.0189
1.50	0.1282	0.1037	0.0814	0.0618	0.0450	0.0313
1.60	0.1686	0.1402	0.1136	0.0894	0.0679	0.0495
1.70	0.2157	0.1839	0.1534	0.1246	0.0983	0.0749
1.80	0.2690	0.2346	0.2007	0.1680	0.1370	0.1084
1.90	0.3275	0.2914	0.2551	0.2191	0.1842	0.1509
2.00	0.3897	0.3530	0.3154	0.2773	0.2393	0.2021
2.10	0.4540	0.4177	0.3800	0.3410	0.3013	0.2614
2.20	0.5186	0.4838	0.4470	0.4084	0.3683	0.3272
2.30	0.5818	0.5493	0.5145	0.4775	0.4384	0.3974
2.40	0.6421	0.6126	0.5806	0.5460	0.5090	0.4697
2.50	0.6983	0.6720	0.6433	0.6120	0.5781	0.5415
2.60	0.7494	0.7266	0.7015	0.6739	0.6436	0.6106
2.70	0.7949	0.7755	0.7541	0.7303	0.7040	0.6749
2.80	0.8345	0.8184	0.8005	0.7804	0.7581	0.7333
2.90	0.8684	0.8552	0.8405	0.8240	0.8055	0.7848
3.00	0.8967	0.8862	0.8744	0.8610	0.8460	0.8292
3.10	0.9200	0.9118	0.9024	0.8919	0.8799	0.8664
3.20	0.9389	0.9325	0.9252	0.9170	0.9077	0.8971
3.30	0.9539	0.9490	0.9435	0.9372	0.9300	0.9219
3.40	0.9657	0.9620	0.9578	0.9531	0.9477	0.9415
3.50	0.9748	0.9720	0.9689	0.9654	0.9614	0.9568
3.60	0.9816	0.9797	0.9774	0.9748	0.9719	0.9685
3.70	0.9868	0.9854	0.9837	0.9819	0.9797	0.9773
3.80	0.9906	0.9896	0.9884	0.9871	0.9856	0.9839
3.90	0.9934	0.9927	0.9919	0.9909	0.9899	0.9887
4.00	0.9954	0.9949	0.9944	0.9937	0.9930	0.9921
4.10	0.9969	0.9965	0.9961	0.9957	0.9952	0.9946
4.20	0.9979	0.9976	0.9974	0.9971	0.9967	0.9963
4.30	0.9986	0.9984	0.9982	0.9980	0.9978	0.9975
4.40	0.9991	0.9989	0.9988	0.9987	0.9985	0.9984
4.50	0.9994	0.9993	0.9992	0.9991	0.9990	0.9989



<u>GRAPH 1</u> Graph of $F_\theta(y) = P[M(\theta) < y]$



GRAPH 2 Graph of the density $f_\theta(y)$ of $M(\theta)$



GRAPH 3 Probability contours of $F_\theta(y)$ as functions of θ and y

THE MAXIMUM OVER AN INTERVAL OF METEOROLOGICAL VARIATES MODELED BY THE STATIONARY GAUSSIAN MARKOV PROCESS

Part II

J. Keilson
University of Rochester
Rochester, New York 14627

H. F. Ross
The Medical School, University of Birmingham
Birmingham, England B15 2TJ

1. INTRODUCTION

It has been observed empirically (Gringorten, 1966, 1968) that fluctuations in upper-level wind velocity, rainfall, temperature, and other meteorological variates may be modeled by a type of random Markov process. A good fit is obtained over a broad range of values when the so-called Ornstein-Uhlenbeck process described below is employed. Acceptance of the validity of such a model permits utilization of its theoretical predictions for long-range operational planning, with on-line predictions available from software. Such software is in development at the U.S.A.F. Geophysical Research Laboratory.

The O-U process is the simplest random process having a stationary distribution whose sample paths are continuous functions of time. It is correspondingly easiest to work with. Certain passage time distributions needed for meteorological planning have been difficult, however, to calculate, and have only recently become available (Keilson and Ross, 1975).

The random variable of central interest to this paper is $M(\theta)$, the maximum of an O-U variate over an interval of duration θ in the absence of any conditioning history. The distribution of the maximum $M(\theta)$ relates directly to the passage time distributions in the Keilson-Ross paper. The calculation technique employed there evaluated zeros and residues in the complex plane of the parabolic cylinder functions, and such techniques again feature in tabulating the distribution of $M(\theta)$.

This paper describes briefly the results obtained for the interval maximum. A more complete and detailed presentation is available (Keilson and Ross, 1978).

It must be emphasized that the O-U model assumes the negligibility of diurnal and seasonal variation. Where such influence cannot be neglected, the results described here cannot be applied without modification.

2. THE O-U PROCESS

The Ornstein-Uhlenbeck process (Cox and Miller, 1965 and Keilson and Ross, 1975) describes wide band fluctuations averaged by exponential smoothing. For the resulting Markov process, $X(t)$, starting at $X(0) = x_0$, one has in the absence of observation subsequent to $t = 0$,

$$E[X(t)] = x_0 e^{-t}\ , \quad t > 0\ ; \tag{1}$$

$$\mathrm{Var}[X(t)] = 1 - e^{-2t}\ , \quad t > 0\ . \tag{2}$$

As time progresses, $X(t)$ settles into a stationary distribution which is standard normal, i.e., one has

$$\lim_{t\to\infty} P[X(t) \le x] = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy\ . \tag{3}$$

Of central interest is the stationary O-U process $X_s(t)$ in which the variate $X(t)$ starts in its stationary normal distribution and keeps this distribution for all time. The maximum $M(\theta)$ is defined in terms of this stationary process by

$$M(\theta) = \max_{t\le t'\le t+\theta} X_s(t')\ , \text{ any } t > 0\ . \tag{4}$$

Its distribution will be designated by

$$F_\theta(y) = P[M(\theta) \le y] = P[X_s(t') \le y;\ t \le t' \le t+\theta]\ .$$

The stationary O-U process is Gaussian, and is sometimes called the Gaussian Markov process. It has for its covariance, when $t_1 < t_2$,

$$\mathrm{cov}[X(t_1), X(t_2)] = e^{-(t_2-t_1)}\ . \tag{5}$$

All correlation is therefore lost when $t_2 - t_1 \gg 1$. The process has a relaxation time or forgetting time (Keilson, 1979)

$$T = 1\ . \tag{6}$$

3. THE MODELING OF METEOROLOGICAL VARIATES

A meteorological process $Z(t)$ modeled via an O-U process $X(t)$ has three parameters:

(a) the mean value m of the associated stationary process $Z_s(t)$;

(b) the standard deviation σ of the associated stationary process $Z_s(t)$;

(c) the relaxation time T of the associated stationary process $Z_s(t)$.

The process Z(t) then relates to the underlying O-U process X(t) by

$$Z(t) = m + \sigma\, X(\tfrac{t}{T}) \quad . \tag{7}$$

These three parameters will vary with geographical location and altitude, apart from diurnal and seasonal changes. The use of these three parameters is required for application of the graphs and tables.* How they are used will be discussed when the graphs are presented.

4. INADEQUACY OF EXTREME VALUE DISTRIBUTIONS

The extreme value distributions of Gnedenko, Gumbel and others are known to be valid for $M(\theta)$, our interval maximum for the O-U process, when θ is sufficiently large, as shown, for example, by Darling and Erdos, 1955. The validity of such asymptotic description has been known to set in very slowly, and the error in the asymptotic distribution of $M(\theta)$ for any value of θ has not been known. The results presented here are accurate for the O-U model. They permit comparison with extreme value prediction and confirm the inadequacy of asymptotic extreme value theory for most interval durations θ of operational planning interest.

5. NUMERICAL RESULTS AND GRAPHS

The distribution of the maximum $M(\theta)$ is directly related to the passage time distribution for the process $X_s(t)$. This arises from the observation that the event that the maximum value of a process sample is less than y in an interval of duration θ is the same as the event that the process sample takes an interval longer than θ to exceed a value y. Consequently, if the passage time distribution of the process is denoted by

$S_y(\theta)$ = P[the time taken for the process to first reach a value y is $\le \theta$]

then

$1 - S_y(\theta)$ = P[that the process does not exceed a value y within a time interval of duration θ]

$= F_\theta(y)$.

Fig. 1 shows $S_y(\theta)$ as a function of θ for various values of y. The time units are relaxation times of the process under consideration in accordance with the discussion in Section 3. Note that at the left edge of the graph the curves do not go to 0 but to values given by the expression

$$S_y(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}}\, dx \quad . \tag{8}$$

This form of presentation of the graph is most useful if the type of question being asked is "given one of the threshold values y = 0.5, 1.0, etc., what is the probability of exceeding that threshold in a time interval of given length?".

Though the graphs terminate at 50 relaxation times, they may easily be extended further since beyond that point they are adequately approximated by exponential curves, which means that values of $S_y(\theta)$ for $\theta > 50$ may be obtained from the equation $S_y(\theta) = 1 - e^{-\lambda(y)\theta}$.

Table I gives the value of $\lambda(y)$ for the set of values of the y parameter used in the graph.

Table I

y	$\lambda(y)$
0.5	0.6488
1.0	0.3882
1.5	0.2087
2.0	0.09727
2.5	0.03768
3.0	0.01161
3.5	0.002752

Fig. 2 shows $F_\theta(y)$ as a function of y for various values of c, where $c = \sqrt{2\log\theta}$. This form of presentation is most appropriate for showing the way in which the y and θ parameters interact for large value of θ (long time durations). The independent variable $c = \sqrt{2\log\theta}$ has been used because it can be shown that in the limit, for large values of θ, the variable c is a good measure of location for the distribution $F_\theta(y)$. This feature can be seen in the graph even for the relatively small values of c used. The relation between c and θ is given explicitly in Table II for the values of c used in the graph.

Table II

c	$\theta = e^{0.5c^2}$
1.0	1.65
1.5	3.08
2.0	7.39
2.5	2.28 x 10
3.0	9.00 x 10
3.5	4.57 x 10^2
4.0	2.98 x 10^3
4.5	2.49 x 10^4
5.0	2.68 x 10^6

For large values of θ, $F_\theta(y)$ approaches the extreme value distribution as a limiting form. This approach is rather slow, however. Numerical calculations indicate that it is not until c reaches a value of about 5 that the error in the distribution function from using the extreme value form is down to approximately 1%. Table II shows that that value of c corresponds to 2.68 x 10^5 relaxation times, which for a relaxation time of one now means a period of about 30 years.

*Tables are given in Keilson and Ross, 1978.

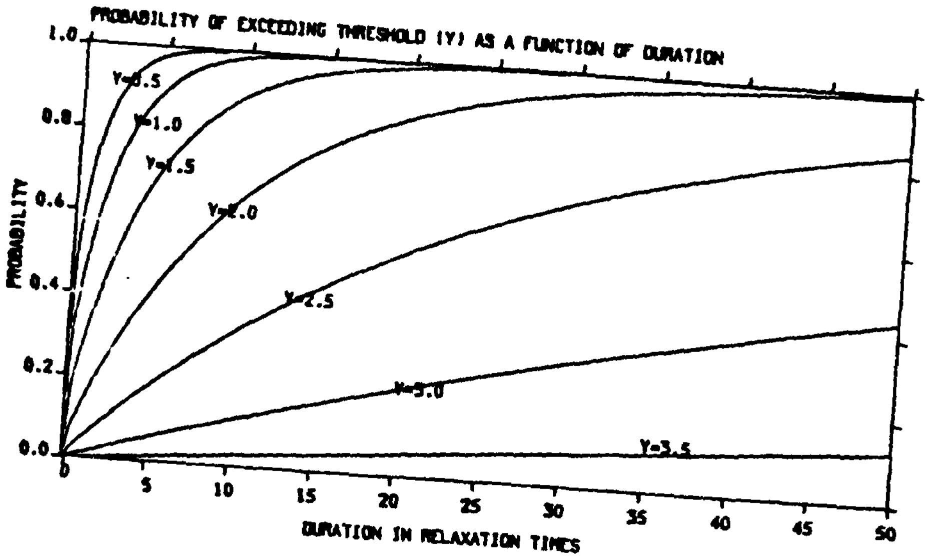


FIG. 1. Probability of exceeding a threshold y within a time interval of specified duration.

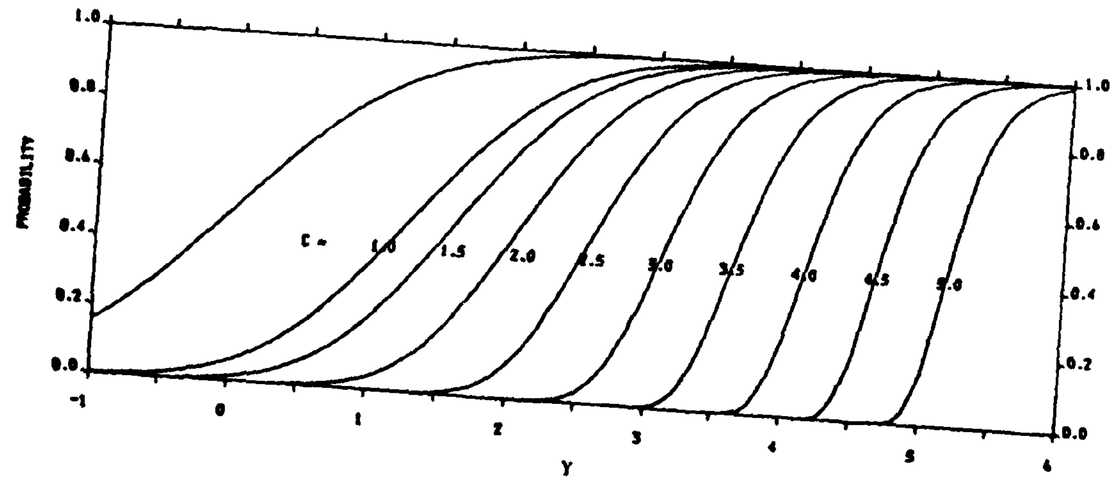


FIG. 2. Probability that the maximum value, $M(\theta)$, of the process is less than y throughout an interval of duration θ. The curve on the extreme left is for $\theta = 0$. The other curves are for the indicated values of c where $c = \sqrt{2\log\theta}$. Values of θ corresponding to the range of c values used are tabulated in Table I.

ACKNOWLEDGMENT

The authors wish to thank I. I. Gringorten for initiating this study and for many helpful discussions. We also wish to thank J. A. Wellner for valuable comments and suggestions, and U. Sumita and Ms. L. Ziegenfuss for technical and editorial contributions.

The support of the U.S.A.F. Geophysical Research Laboratory under contract number F19628-78-C-0158 is gratefully acknowledged.

REFERENCES

Cox, D.R. and Miller, H. D. (1965). The Theory of Stochastic Processes, John Wiley, New York.

Darling, D. A. and Erdos, P. (1955). "A limit theorem for the maximum of normalized sums of independent random variables," Ann. Math. Stat., 26, pp. 294-300.

Gringorten, I. I. (1966). "A stochastic model of the frequency and duration of weather events," J. App. Meteorology, 5, pp. 606-624.

Gringorten, I. I. (1968). "Estimating finite-time maxima and minima of a stationary Gaussian Ornstein-Uhlenbeck process by Monte Carlo simulation," Amer. Stat. Assn. J., pp. 1517-21.

Keilson, J. (1979). Markov Chain Models - Rarity and Exponentiality, Springer-Verlag, New York.

Keilson, J. and Ross, H. F. (1975). "Passage time distributions for Gaussian Markov statistical processes," Selected Tables in Mathematical Statistics, Vol. III (edited by the Institute of Mathematical Statistics). American Math. Society, Providence, Rhode Island.

Keilson, J. and Ross, H. F. (1978). "The maximum of the stationary Gaussian Markov process over an interval," Graduate School of Management, University of Rochester, Working Paper Series No. 7910.